IIT JAM Real Test

Mathematical Statistics

Q1 Let J denote the matrix of order $n \times n$ with all entries 1 and let B be a $(3n) \times (3n)$ matrix given by

$$B = \begin{pmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{pmatrix}$$

Then the rank of B is

(A) $2n-1$ (B) $2n$

(C) 3 (D) 1

Q2 Consider the following system of equations:

$x_1 + x_2 - x_3 = 9$

$2x_1 + 3x_3 = a$

$3x_1 + x_2 + 2x_3 = 18$

what should a be so that this system is consistent?

(A) 10 (B) 1

(C) 2 (D) 9

Q3 Let A be a 3×4 matrix and B be a 4×3 matrix with real entries such that A B is non-singular. Consider the following statements:

P: Nullity of $\boldsymbol{A}$ is $\mathbf{0}$.

Q: BA is a non-singular matrix.

Then

(A) Both P and Q are TRUE

(B) P is TRUE and Q is FALSE

(C) P is FALSE and Q is TRUE

(D) Both P and Q are FALSE

Q4

Suppose that X is a continuous random variable with cumulative distribution function F, density f, and moment generating f

$\frac{M(t)}{M(-t)} = e^{6t}, \quad \forall\, t \in (-h, h),\ h > 0$

then which of the following statements is FALSE?

(A) $P(X-3 = 3-X) = 0$

(B) $f(x-3) = f(3-x)$ for all $x \in \mathbb{R}$

(C) 3 is the median of X.

(D) $X-3$ and $3-X$ are identically distributed

Q5

If X_1 and X_2 are two independent and identically distributed random variables having the probability density function $f\left(x\right) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R},$

then the probability density function of $U = X_1 X_2$ is

(A) $f_U\left(u\right) = \frac{1}{\pi^2(1+u^2)}, \quad u \in \mathbb{R}$

(B) $f_U\left(u\right) = \frac{2\log|u|}{\pi^2(u^2-1)}, \quad u \in \mathbb{R}$

(C) $f_U\left(u\right) = \frac{\log|u|}{\pi(u^2-1)}, \quad u \in \mathbb{R}$

(D) $f_U\left(u\right) = \frac{2\log u}{\pi(u+1)(u-1)}, \quad u \in \mathbb{R}$

Q6 Let $X_1, X_2, \ldots$ be i.i.d. N(0,1) random variables. Define

$$Z_n = \frac{\sum_{i=1}^{n} \frac{X_{2i-1}}{X_{2i}}}{\sum_{i=1}^{2n} X_i^2}.$$

Then the limiting distribution of Z_n, as $n \to \infty$, is

(A) $N(0,1)$ (B) $N(0,2)$

(C) $C(0,1)$ (D) $C\left(0, \frac{1}{2}\right)$

Q7 Consider the probability space $(\Omega, \mathscr{F}, P_U)$, where $\Omega = (0,1), \mathscr{F} = \mathscr{B}(0,1)$, and $P_U((0,x]) = x,\ x \in (0,1)$.

Let X be function define as $X\left(\omega\right) = \log\frac{1}{1-\omega}, \quad \omega \in \left(0,1\right)$.

Then the value of $\mathrm{Var}\left(e^{-\frac{3}{4}X}\right)$ is

(A) $\frac{1}{45}$ (B) 18

(C) $\frac{18}{245}$ (D) $\frac{16}{245}$

Q8 Suppose X~Uniform$(2,12)$. Define

$$Z = \begin{cases} X-4, & \text{if } X \le 8, \\ X+2, & \text{if } X > 8. \end{cases}$$

Then $E(Z)$ is

(A) 3.6 (B) 5.4

(C) 7.8 (D) 9.4

Q9 Suppose that the pdf of the random variable X is

$$f(x) = \begin{cases} \frac{3x^2}{\theta^3}, & 0 \le x \le \theta, \\ 0, & \text{otherwise.} \end{cases}$$

where $\theta > 0$ is an unknown parameter. Based on a single observation X, the confi

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$

(C) $\frac{3}{4}$ (D) $\frac{7}{8}$

Q10 Consider a distribution with probability mass function

$$f(x \mid \theta) = \begin{cases} \frac{\theta}{3}, & x = 0, \\ \frac{2(1-\theta)}{3}, & x = 1, \\ \frac{\theta+1}{3}, & x = 2, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } 0 < \theta < 1.$$

In a random sample of size $n = 120$ from this distribution, the observed counts of 0, 1, 2 are 40, 40 and 40 respectively.

Then the maximum likelihood estimate of θ based on the observed samp

(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{2}{\sqrt{3}}$

(C) $\frac{3}{4}$ (D) $\frac{1}{4}$

Q11 Let $f: \mathbb{R} \to \mathbb{R}$ be defined by,

$f\left(x\right) = x^3 + x + a$, where $a \in \mathbb{R}$. Then:

(A) $f(x)$ has no real root for any a
(B) $f(x)$ has one real root if $a \geq 0$ and three real roots if $a < 0$
(C) $f(x)$ has exactly one real root for all a
(D) $f(x)$ has three real roots for all values of a

Q12 Let $f(x) = x^{1+\sin x}$, then $\lim_{x\to\infty} x^{1+\sin(x)}$
(A) Does not exist
(B) Exist and equal to zero
(C) Exist infinitely
(D) Exist but non-zero and finite

Q13 Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ and *I* be the 3×3 identity matrix. If $6A^{-1} = aA^2 + bA + cI$ for $a,\ b,\ c \in \mathbb{R}$ then (a, b, c) equals
(A) (1,2,1) (B) $(1, -1, 2)$
(C) (4,1,1) (D) (1,4,1)

Q14 Consider a sequence $\{x_n\}$ defined as $x_n = \frac{[\alpha]+[2\alpha]+\ldots+[n\alpha]}{n^2}$, where [.] is the greater integer function, then
(A) $\lim_{n\to\infty}$ x_n may or may not exist depending on α
(B) $\lim_{n\to\infty} x_n = \alpha$
(C) $\lim_{n\to\infty} x_n = \frac{\alpha}{2}$
(D) $\lim_{n\to\infty} x_n = 0$

Q15 Let $<a_n>$ be a sequence of real numbers defined as $a_{n+1} = \frac{a_n}{1+a_n^2}; a_1 \in (0, \infty)$ then
(A) $<a_n>$ is monotonic but not bounded
(B) $<a_n>$ is bounded but not monotonic
(C) $<a_n>$ is increasing and bounded
(D) $<a_n>$ is decreasing and bounded

Q16 Define
$f(x) = \lim_{n\to\infty}\left[\lim_{t\to 0} \frac{\sin^2(n!\pi x)}{\sin^2(n!\pi x)+t^2}\right]; x \in \mathbb{R}$ then
(A) $f(x)$ is continuous only on $x \in \mathbb{Q}$
(B) $f(x)$ is continuous only on $x \in \mathbb{Q}^c$
(C) $f(x)$ is continuous for all $x \in \mathbb{R}$
(D) $f(x)$ is nowhere continuous on $\mathbb{R}$

Q17 Let $X_1, X_2,$ $\ldots$ be a sequence of i.i.d. random variables having the probability density function
$$f(x) = \frac{1}{2}e^{-|x|}, \quad -\infty < x < \infty,$$
Consider $S_n = [|X_1|] + [|X_2|] + \cdots + [|X_n|]$.
Where $|t| = \begin{cases} t, & t \geq 0, \\ -t, & t < 0 \end{cases}$ and $[t]$ denotes the greatest integer less than equals to t.
Suppose that there exist sequences a_n and $b_n > 0$ such that $\frac{S_n - a_n}{b_n} \xrightarrow{d} N(0,1)$ as $n \to \infty$.

Then the value of $\lim_{n\to\infty} \frac{1}{2}\sqrt{n}\log\left(1 - \frac{b_n}{a_n}\right)$ is
(A) 0 (B) $-\frac{\sqrt{e}}{2}$
(C) $-\frac{1}{2}\log e$ (D) $-\sqrt{e}\log e$

Q18 Let $X_1, \ldots, X_n$ be a random sample from
$$f_\theta(x) = \begin{cases} \frac{\theta}{x^2}, & x \geq \theta, \\ 0, & \text{otherwise}, \end{cases} \quad \theta > 0.$$
Let $x_{(1)} = \min(x_1, \ldots, x_n)$.
Critical region of MP size α test for testing $H_0 : \theta = 1$ against $H_A : \theta = 2$ is given by
(A) $\{(x_1, \ldots, x_n) : x_{(1)} > \alpha^{-1/n}/2\}$
(B) $\{(x_1, \ldots, x_n) : x_{(1)} > (1 - \alpha^{-1/n})\}$
(C) $\{(x_1, \ldots, x_n) : x_{(1)} > \alpha^{-1/n}\}$
(D) $\{(x_1, \ldots, x_n) : x_{(1)} > \alpha^{1/n}\}$

Q19 Let (X, Y) be a continuous random vector with joint pdf
$$f(x,y) = \begin{cases} k|x|, & 0 \leq x \leq \frac{1}{3},\ x^2 \leq y \leq 5x^2, \\ 0, & \text{otherwise}. \end{cases}$$
where $k > 0$ is a constant.
Then the value of $P(2X + 3Y > 1)$ is
(A) $\frac{244}{375}$ (B) $\frac{131}{375}$
(C) $\frac{141}{375}$ (D) $\frac{234}{375}$

Q20 Suppose X_1 and X_2 are independent Bernoulli $\left(\frac{1}{2}\right)$ random variables. If $M = \min(X_1, X_2)$, then the value of $\operatorname{Corr}(X_2, M)$ is
(A) $\frac{1}{2}$ (B) $\frac{2}{\sqrt{3}}$
(C) $\frac{1}{\sqrt{3}}$ (D) $\frac{1}{4}$

Q21 Trains depart from a railway station in accordance with a Poisson process with rate 80 trains per hour. 10% of these trains are freight trains and 90% are passenger trains.
Given that 16 freight trains have departed during one hour, what is the expected total number of trains that have departed in that hour?
(A) 64 (B) 80
(C) 96 (D) 100

Q22 Let X, Y, Z be independent and identically distributed $N(0, 1)$ random variables.

Define $U = X + Y, \quad V = 4X - 2Z, \quad W = 3Y + Z$.
Then the partial correlation coefficient of V and W, given U is.
(A) $-\frac{1}{2}$ (B) 0
(C) $\frac{8}{\sqrt{66}}$ (D) $-\frac{4}{\sqrt{33}}$

Q23

Let X_1 , X_2 , $\cdots$, X_n be random sample of size n (≥ 2) from a distribution having probability density function

$$f(x,\ \theta) = \begin{cases} \frac{1}{3}e^{-\frac{(x-\theta)}{3}}, & x > \theta \\ 0, & \text{otherwise} \end{cases}$$

where $\theta \in \mathbb{R}$. Let $X_{(1)} = \min(X_1,\ X_2,\ \cdots,\ X_n)$. Then $\mathbb{E}\left(X_1 + e^{|X_1 - X_3|} - e^{|X_2 - X_4|} \middle| X_{(1)}\right)$ equals

(A) $\frac{nX_{(1)}+3}{n^2}$ (B) $X_{(1)} - \frac{3}{n}$

(C) $\frac{n(X_{(1)}+3)-3}{n}$ (D) $\frac{X_{(1)}}{n^2} + 3$

Q24 The random variable X takes the values $\{-1,\ 0,\ 1\}$according to the distribution $\{F_p : 0 < p < \frac{1}{3}\}$ such that X

$$= \begin{cases} -1 & \text{with probability} \quad p, \\ 0 & \text{with probability} \quad 2p, \\ 1 & \text{with probability} \quad 1-3p. \end{cases}$$

Let X_1 and X_2 be random sample from this distribution. Define $T = |X_1 + X_2|$

Then , among statistics X and T

(A) Both X and T are complete.

(B) X is complete but T is NOT complete .

(C) Both X and T are NOT complete.

(D) X is NOT complete but T is complete .

Q25 Suppose X be $N(0,9)$ and Y is independent of X with propbability mass function

$$P(X = k) = \begin{cases} \frac{1}{2}, & k = -1 \\ \frac{1}{2}, & k = 1 \end{cases}$$

Define $Z = XY + \frac{X}{Y}$, then $P\left(Z^3 - 2Z^2 - Z + 2 < 0\right)$ is equal to

(A) $\Phi\left(-\frac{1}{6}\right) - \Phi\left(\frac{1}{3}\right)$

(B) $2\Phi\left(-\frac{1}{6}\right) + \Phi\left(\frac{1}{3}\right) - 1$

(C) $1 + \Phi\left(\frac{1}{3}\right) - \Phi\left(\frac{1}{6}\right)$

(D) $1 + \Phi\left(\frac{1}{3}\right) - 2\Phi\left(\frac{1}{6}\right)$

Q26 Let X_1 , X_2 , X_3 , X_4 , X_5 be a random sample of size 5 from an $N(\theta, 5)$ distribution

Where $\theta \in \mathbb{R}$ is an unknown parameter . Let $\overline{X} = \frac{1}{5}\sum_{i=1}^{5} X_i$, $g(\theta) = e^{2\theta}$and $L(\theta)$ be the Cramer-Rao lower bound for variance of unbiased estimator of $g(\theta)$. Then which of the following statement is FALSE?

(A) $\overline{X} + e^{2\left(\overline{X}-1\right)}$ is complete sufficient for θ.

(B) $L(\theta) = 4e^{4\theta}$

(C) $e^{2\left(\overline{X}-1\right)}$ is the uniformly minimum variance unbiased estimator of $g(\theta)$.

(D) The variance of the undiased estimator of $g(\theta)$ attains the Cramer -Rao Lower bound.

Q27 Let $Z_1, Z_2,$ ... be i.i.d. random variables with discrete uniform distributionover {1,

Let X_n be the remainder when Y_n is divided by 2026 .

Then, which of the following statements is true?

(A) $\lim_{n\to\infty} P\left(X_n = 2025\right) = \frac{1}{2024}$

(B) $\lim_{n\to\infty} P\left(X_n = 2025\right) = \frac{1}{2025}$

(C) $\lim_{n\to\infty} P\left(X_n = 2025\right) = \frac{1}{2026}$

(D) $\lim_{n\to\infty} P\left(X_n = 2025\right) = 0$

Q28 Consider the simple linear model $Y_i = \beta i^2 + \varepsilon_i, \quad i = 1, \ldots, 100,$ where $\varepsilon_i \sim N\left(0, \sigma^2\right)$ are i.i.d. random variables.

Define $\widehat{\beta}_1 = \frac{\sum_{i=1}^{100} i^2 Y_i}{\sum_{i=1}^{100} i^4}, \quad \widehat{\beta}_2 = \frac{\sum_{i=1}^{100} Y_i}{\sum_{i=1}^{100} i^2}.$

Which of the following statement is FALSE ?

(A) $\widehat{\beta}_1$ is unbiased estimator for β.

(B) $\widehat{\beta}_2$ is unbiased estimator for β.

(C) $\widehat{\beta}_1$ has larger variance than $\widehat{\beta}_2$.

(D) $\widehat{\beta}_2$ has larger variance than $\widehat{\beta}_1$.

Q29 Let $\boldsymbol{X} = (X_1,\ X_2,\ X_3,\ X_4)^t$ be random vector having the probability mass function

$$f_{\mathbf{X}}\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{cases} \frac{c}{x_1!\,x_2!x_3!x_4!} & \text{if } x_1,\ x_2,\ x_3,\ x_4 \in \{0, 1, \cdots, 50\},\ x_1 + x_2 + x_3 + x_4 = 50 \\ 0, & \text{otherwise,} \end{cases}$$

where c is a real constant.

Suppose $\mathbf{\Sigma}_{4\times 4}$ be the variance covariance matrix of random vector $\boldsymbol{X}$. Then which of the following statement is FALSE.

(A) rank $(\mathbf{\Sigma}) = 3$

(B) trace $(\mathbf{\Sigma}) = \frac{75}{2}$

(C) The largest eigenvalue of $\mathbf{\Sigma}$ is $\frac{25}{4}$.

(D) 1 is always an eigenvalue of $2\mathbf{\Sigma}^2 - \mathbf{\Sigma} + \mathbf{I}$.

Q30 Let $X, Y,$ Z be independent and identically distributed (i.i.d.) random variables, each following Bernoulli(θ), where $0 < \theta < 1$.

Then, which of the following statement is NOT True ?

(A)

$\left(79e^X + 2024e^Y,\right.$
$\left.23\log(1+Z)\right)$ is a sufficient statistic for θ

(B) e^{X+Y+Z} is a minimal sufficient statistic for θ

(C) $13(X+Y+Z)^2$ is a complete sufficient statistic for θ

(D) $\log(X-2Y+Z)$ is an ancillary statistic.

Q31 Consider the function

$$f(x) = \begin{cases} \frac{\sin 1/q}{\sin 1/p}, & x = p/q \\ 1+\sin(x-1), & x \in \mathbb{Q}^c \cup \{0\} \end{cases}$$

Let $S = \left\{c \in \mathbb{R} : \lim_{x\to c} f(x) \text{ exists}\right\}$. Then

(A) $|S| \geq 1$

(B) S is finite

(C) S is countably infinite

(D) S is empty

Q32 Let V be a vector space over $\mathbb{F}$ with dimension n. Let $T: V \to V$ be a linear transformation with only 1 as eigenvalue. Then which of the following must be true?

(A) $T - I = 0$ (B) $(T-I)^{n-1} = 0$

(C) $(T-I)^n = 0$ (D) $(T-I)^{2n} = 0$

Q33 If $M = \begin{bmatrix} 2 & 0 & 3 & 2 & 0 & -2 \\ 0 & 1 & 0 & -1 & 3 & 4 \\ 0 & 0 & 1 & 0 & 4 & 4 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$,

$b_1 = \begin{bmatrix} 5 \\ 1 \\ 1 \\ 4 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 3 \end{bmatrix}$.

Then which of the following are true ?

(A) Both systems $MX = b_1$ and $MX = b_2$ are inconsistent

(B) Both systems $MX = b_1$ and $MX = b_2$ are consistent

(C) The system is $MX = b_1 - b_2$ consistent

(D) The systems is $MX = b_1 - b_2$ inconsistent

Q34 Consider a trivariate discrete random variable $\mathbf{W} = (X, Y, Z)^t$ with pmf

$$f_{\mathbf{W}}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \left(\tfrac{1}{3}\right)^x \left(\tfrac{1}{6}\right)^y \left(\tfrac{1}{2}\right)^z \text{ with } x,\ y,\ z \in \{0,1\}$$

If we take random sample $w_1, w_2, \cdots, w_{10}$ from $\mathbf{W}$. Define

$$S_{10} = \frac{1}{10}\sum_{i=1}^{10}(w_i - \overline{w})(w_i - \overline{w})^{\mathrm{T}}$$

then which of the following statements is / are TRUE?

(A) $1 - X$~Ber $\left(\frac{2}{3}\right)$

(B) Var $(Y^4) = \frac{5}{36}$

(C) trace $(\mathbb{E}(S_{10})) = \frac{110}{63}$

(D) $\mathbb{E}\left(\binom{100}{Z}\right) = 50$

Q35 Let $U_1, U_2, \ldots$ be a sequence of independent and identically distributed $U(0,1)$ ran
$G_n = \left(\prod_{i=1}^n U_i\right)^{1/n}$
be the geometric mean of $U_1, U_2, \ldots, U_n$ for $n \in \mathbb{N}$.
Let X_1 and X_2 be degenerate random variables such that $P(X_1 = 0) =$
Then, which of the following statements are true?

(A) G_n converges in r-th mean to X_1 as $n \to \infty$, for any $r > 0$.

(B) G_n converges in probability to X_1 as $n \to \infty$.

(C) G_n converges in distribution to X_2 as $n \to \infty$.

(D) G_n converges in r-th mean to X_2 as $n \to \infty$, for any $r > 0$.

Q36 Let $X_1, X_2, \cdots, X_n$ be random sample of size n having probability density function $f(x|\theta)$.
Then which of the following statements is (are) TRUE ?

(A) If $|T|$ is a sufficient statistic for $\theta \in \Theta$, then T is also a sufficient statistic for $\theta \in \Theta$.

(B) If T is a complete statistic, then $e^T - T - 1$ is also a complete statistic.

(C) If T is a complete statistic and T^2 is a sufficient statistic, then T is a

(D) If T is a minimal sufficient statistic, then $T^3 - 2T^2 + 2T + 1$ is also a minimal sufficient statistic.

Q37 Suppose that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of independent and identically distributedrandom va
probability density function $f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$
Then which of the following statements is (are) TRUE?

(A) $\frac{X_1 + 2\sqrt{3}X_2}{\sqrt{3}}$~Cauchy$\left(0,\ 2 + \frac{1}{\sqrt{3}}\right)$.

(B) $\frac{1}{n}\sum_{i=1}^n X_i$ converges to 0 in probability, as $n \to \infty$.

(C) The median of $\{X_1, X_2, \ldots, X_{2n+1}\}$ converges to 0 in probability, as $n \to \infty$.

(D) $E\left(|X_1|^{1/4}\right) = \frac{1}{\sin(5\pi/8)}$.

Q38

Let X_1 , X_2 and X_3 be independent $N(0,1)$ random variables. Define

$$\operatorname{sgn}(u) = \begin{cases} -1, & \text{if } u < 0 \\ 0, & \text{if } u = 0, \\ 1, & \text{if } u > 0 \end{cases}$$

Let $Y_1 = X_1 \operatorname{sgn}(X_2)$, $Y_2 = X_2 \operatorname{sgn}(X_3)$ and $Y_3 = X_3 \operatorname{sgn}(X_1)$

Then which of the following statements is (are) TRUE ?

(A) $Y_3 \sim N(0,1)$

(B) $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

(C) $P(Y_1 > 0, Y_2 < 0, Y_3 > 0) = \frac{1}{4}$

(D) $P(Y_1 < 0, Y_2 > 0, Y_3 < 0) = \frac{1}{4}$

Q39 Let $(Z_1, W_1), (Z_2, W_2), \ldots, (Z_{12}, W_{12})$ be a random sample from a bivariate normal distribution $BVN(\lambda_1, \lambda_2, \tau_1^2, \tau_2^2, \rho)$ with $\lambda_1 = 4,\ \lambda_2 = 7,\ \tau_1^2 = 9,\ \tau_2^2 = 16,\ \rho = \frac{1}{3}$.

Which of the following statements are true?

(A) The distribution of $\frac{1}{\sqrt{12}} \sum_{i=1}^{12} \left(Z_i - 4\right)$ is $N\left(0, 9\right)$.

(B) The distribution of $\frac{1}{11} \sum_{i=1}^{12} \left(Z_i + W_i - 11\right)^2$ is $3\chi_{12}^2$.

(C) The distribution of $\frac{\sqrt{11}\,(Z_1 - 4)}{\sqrt{\sum_{i=2}^{12}\left(Z_i - 4\right)^2}}$ is t_{11}.

(D) The distribution of $\frac{\sum_{i=1}^{4}\left(W_i - 7\right)^2/4}{\sum_{i=5}^{12}\left(W_i - 7\right)^2/8}$ is $F_{4,8}$

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Q40 Let X be a random sample of size one from the probability density function

$$f\left(x \mid \theta\right) = \begin{cases} 2\theta\, e^{-2\theta(x-1)}, & x > 1, \\ 0, & \text{otherwise,} \end{cases} \quad \theta > 0.$$

Suppose we want to test the null hypothesis $H_0 : \theta = 2$ against the alternative hypothesis $H_1 : \theta \neq 2$, based on the observed value x of X.

Then, which of the following statements are true?

(A) The likelihood function is maximized at $\theta = \frac{1}{2(x-1)}$.

(B) The maximum value of the likelihood function is $\frac{2e^{-1}}{x-1}$.

(C) The likelihood ratio test for testing H_0 against H_1 rejects H_0 if $2(x-1)e^{-2x} < k$, for some $k > 0$.

(D) The likelihood ratio test for testing H_0 against H_1 rejects H_0 if $x > c$, for some $c > 2$.

Q41 Let $a_n = \left(2^n + n2^n \sin^2 \frac{n}{2}\right)^{\frac{1}{2n - n\cos\frac{1}{n}}}$ then $\lim_{n\to\infty} a_n$ is

Q42 Consider $<a_n>$ be a sequence of real numbers defined as $a_n = \frac{n^5 \cdot n!}{5\cdot 6\cdot 7\cdot \ldots \cdot (5+n)}$, then $\lim_{n\to\infty} a_n$

Q43 Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f\left(x\right) = \begin{cases} x^3 \sin\frac{1}{x} & x \in \mathbb{Q} \\ x^2 & x \in \mathbb{Q}^c \end{cases}$ .Then number of points at which f is continuous is

Q44 Let $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} 1 & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & 1 \end{pmatrix}\right)$

Then the value of $\operatorname{Var}(X_1 + 2X_2 \mid 3X_1 - X_2)$ is ____________ (answer correct to two decimal places).

Q45 Let X be a random variable with cumulative distribution function $F(x)$

$$F(x) = \begin{cases} 0, & x < -1, \\ \frac{x+1}{4}, & -1 \le x < 0, \\ \frac{x+1}{3}, & 0 \le x < 1, \\ \frac{x+3}{6}, & 1 \le x < 2, \\ 1, & x \ge 2. \end{cases}$$

Then the value of $\mathbb{E}\left(e^{2X}\right)$ is ____________. (round off to two decimal places)

Q46 Let $X_1, \ldots, X_{15}$ and $Y_1, \ldots, Y_{30}$ be independent random variables with common density

$f\left(x\right) = \frac{2}{\sqrt{2\pi}\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x \ge 0.$

Suppose the observed values of $\sum_{i=1}^{15} X_i^2$ and $\sum_{j=1}^{30} Y_j^2$ are 150 and 255 respectively.

Then the maximum likelihood estimate of σ is equal to__________.

Q47 Let $(X_i, Y_i),\ i = 1, \ldots, n,$ be independent pairs from the density

$f\left(x, y\right) = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left[(x - \rho\cos\theta)^2 + (y - \rho\sin\theta)^2\right]\right\}, \quad 0 \le \theta \le 2\pi, \ , \ (x, y) \in \mathbb{R}\times\mathbb{R}$

where ρ is known.

Suppose the observe values of $\sum_{i=1}^{n} X_i$ and $\sum_{i=1}^{n} Y_i$ are $100\sqrt{3}$ and 300 respectively .

if α be the method of moments estimate of. θ then $\frac{27\alpha}{\pi}$ is equal to _____

Q48 Let X_1 , X_2 , $\cdots$, X_5 be random sample of size 5 from $N(\mu, 1)$ distribution where μ is unknown

. Then the value of uniformly minimum variance unbiased estimator of

Q49 Let $X_{(1)}, X_{(2)}$ denote the order statistics of a random sample of size 2 from $\mathrm{N}(\mu, 2)$.

Q50 Let $X_1, X_2, \ldots, X_8$ be a random sample from a distribution with probability density function

$f_X\left(x\right) = \begin{cases} c(\lfloor x \rfloor + 1), & 0 < x < 3, \\ 0, & \text{otherwise}, \end{cases}$
where c is a real constant.

where $\lfloor x \rfloor$
denotes the greatest integer less than or equal to.
x.
Let $F_8(x)$
be the empirical distribution function of the sample. If
α is the variance of $F_8(2)$,
then 32α (in integer) is equal to
_____________.

Q51 If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then Trace $\left(A^{50}\right)$ is

Q52 Let $k \in \mathbb{R}$, if

$\lim_{x \to 0^+} \left(\sin kx + \cos kx + x\right)^{\frac{4}{x}} = e^{12}$, then

the value of k is

Q53 The value of the double integral
$\iint_D \frac{xy}{\sqrt{1-y^2}} dxdy$ over the first quadrant of the circle $x^2 + y^2 = 1$ is

Q54 Let X be a discrete random variable with p.m.f.
$f \in \{f_0, f_1\}$, where

	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$
$f_0(x)$	0.1	0.1	0.1	0.1	0.6
$f_1(x)$	0.05	0.06	0.08	0.09	0.72

The power of most powerful size $\alpha = 0.2$ test for testing $H_0 : X \sim f_0$ against $H_A : X \sim f_1$, based on X, is _________
(exact two decimal places)

Q55 Let $X_1, \cdots,$

X_{15} be random sample from the probability density function $f(x|\theta) = \begin{cases} \frac{4x^3}{\theta} e^{-\frac{x^4}{\theta}} & , x > 0 \\ 0 & , x \le 0 \end{cases}$ Consider the testing of hy

(rounded off to 3 decimal places)

Q56 Suppose X and U
are independent random variable with
$P(X = k) = \frac{1}{2027}$, $k = 0, 1, 2,$
$\cdots, 2026$
and U having a uniform distribution on
$[0, 1]$. Then the value of
$512\, e^2$
$\mathbb{P}(-\log(X + U - [X + U] > 2))$
is equal to ______________.
(Answer in integer)
where $[t]$
denote greatest integer function less than equal to
t.

Q57

In a random experiment, a fair coin is tossed once. Then, an unbiased si
$S = \begin{cases} 120, & \text{if Head appears,} \\ 121, & \text{if Tail appears.} \end{cases}$
Let X denote the total number of times 3 appears out of S.
If p is the $P(\text{Head} \mid X = 30)$
then the value of $1151p$ is equal to
______________.
(Answer in integer)

Q58 Let
A
$= \left\{(x, y) \in \mathbb{R}^2 : \ |x| - \frac{1}{3} < y < |x| + \frac{1}{3}\right\}$.
Let the joint probability density function of $(X,$
$Y)$ be
$f(x, y) = \begin{cases} \frac{1}{2} e^{-\frac{2}{3}|x-1|}, & (x, y) \in A, \\ 0, & \text{otherwise}, \end{cases}$
If , the covariance between the random variables X and Y is equal to α -

Q59 Let X and Y be independent random variables such that
X follows $U(0, 2)$ distribution and
Y follows Bernoulli distribution with success probability $\theta \in (0, 1)$.
Define $W = X + Y$.
Let $w_1 = 0.3,\ w_2 = 2.4,\ w_3 = 0.8,\ w_4 = 1.9,\ w_5 = 2.1,\ w_6 = 0.6$
be the observed values from the distribution of W
.
Then the maximum likelihood estimate of $90\theta^2$ equals ___________ .

Q60 Let $\{Y_n\}_{n \ge 1}$

be sequence of independent and identically distributed random variabl

Answer Key

Q1	(C)	Q31	(A, B)
Q2	(D)	Q32	(C, D)
Q3	(D)	Q33	(A, C)
Q4	(B)	Q34	(A, B, C)
Q5	(B)	Q35	(C, D)
Q6	(D)	Q36	(A, B, C, D)
Q7	(C)	Q37	(A, C, D)
Q8	(B)	Q38	(A, B, D)
Q9	(D)	Q39	(A, B, C)
Q10	(A)	Q40	(A, C)
Q11	(C)	Q41	2~2
Q12	(A)	Q42	24~24
Q13	(D)	Q43	1~1
Q14	(C)	Q44	5.29~5.39
Q15	(D)	Q45	14.24~14.32
Q16	(D)	Q46	3~3
Q17	(B)	Q47	9~9
Q18	(C)	Q48	0.11~0.17
Q19	(A)	Q49	0.5~0.5
Q20	(C)	Q50	1~1
Q21	(B)	Q51	3~3
Q22	(C)	Q52	2~2
Q23	(B)	Q53	0.12~0.18
Q24	(D)	Q54	0.24~0.24
Q25	(D)	Q55	0.02~0.029
Q26	(D)	Q56	512~512
Q27	(C)	Q57	546~546
Q28	(C)	Q58	5~5
Q29	(C)	Q59	10~10
Q30	(D)	Q60	3.86~3.93

Hints & Solutions

Q1 Text Solution:

Given $B=\begin{pmatrix}0&0&J\\0&J&0\\J&0&0\end{pmatrix}_{3n\times 3n}$

where $J=[a_{ij}=1]_{n\times n}$

Consider $n=1 \Rightarrow J=(1)_{1\times 1}$

$\Rightarrow B=\begin{pmatrix}0&0&(1)\\0&(1)&0\\(1)&0&0\end{pmatrix}$

$\Rightarrow \rho(B)=3$

Hence, options (A),(B) and (D) are discarded.

Therefore option (C) is correct.

Also we can prove the option (C)

$B=\begin{bmatrix}0&0&J\\0&J&0\\J&0&0\end{bmatrix}$

After applying finitely many row operations, we get

$B\sim\begin{bmatrix}J&0&0\\0&J&0\\0&0&J\end{bmatrix}$

which is a block diagonal matrix.

Now, using the fact that the rank of a block diagonal matrix

is equal to the sum of rank of each individual diagonal block, we get

$\rho(B)=\rho(J)+\rho(J)+\rho(J)=3\cdot\rho(J)=3$

$.1=3$

Q2 Text Solution:

Consider the augmented matrix

$[A|b]=\left[\begin{array}{ccc|c}1&1&-1&9\\2&0&3&a\\3&1&2&18\end{array}\right]$

$\sim\left[\begin{array}{ccc|c}1&1&-1&9\\0&-2&5&a-18\\0&0&0&-9-a+18\end{array}\right]$

$\rho[A]=\rho(A|b)$ if

$-9-a+18=0$ i.e. a = 9

Q3 Text Solution:

As AB is a non-singular 3×3, matrix so

$\rho(AB)=3.$

Also $\rho(AB)\le\min\{\rho(A),\rho(B)\}$

$3\le\rho(A)$

Here A is 3×4 matrix, hence $\rho(A)\le 3$

$\Rightarrow \rho(A)=3$

Similarly $\rho(B)=3$

and nullity of A is 4-3=1,

and statement P is false.

and $\rho(BA)\le\min\{\rho(A),\rho(B)\}$

$\rho(BA)\le 3$

So BA is singular, and statement Q is false.

Hence option (d) is correct.

Q4 Text Solution:

Solution.

Given a continuous random variable X with MGF M (t) satisfying $\dfrac{M(t)}{M(-t)}=e^{6t},\ \forall t\ge 0.$

Define $Y=X-3$. Then $M_Y(t)=E(e^{tY})$ $=E(e^{t(X-3)})=e^{-3t}M(t).$

$M_Y(-t)=e^{3t}M(-t)\quad\Rightarrow$
$\dfrac{M_Y(t)}{M_Y(-t)}=\dfrac{e^{-3t}M(t)}{e^{3t}M(-t)}=e^{-6t}\dfrac{M(t)}{M(-t)}$
$=e^{-6t}\cdot e^{6t}=1.$

$\Rightarrow\ M_Y(t)=M_Y(-t)\ \forall t\quad\Rightarrow\ Y$ and $-Y$ have the same MGF, hence the same distribution.

Thus $X-3$ and $3-X$ are identically distributed.

$\Rightarrow$ the distribution of X is symmetric about 3.

(D) is therefore TRUE.

If a continuous distribution is symmetric about 3,

then $P(X\le 3)=P(X\ge 3)=\dfrac{1}{2}$, so 3 is a median.

Hence (C) is TRUE.

Further, for symmetry about 3 we have $f(3+u)=f(3-u)\ \forall u.$

Let $x=3+u$. Then $f(x)=f(6-x)\ \forall x.$

Thus the correct symmetry relation for the pdf is $f(x)=f(6-x)$, not $f(x-3)=f(3-x)$.

Note that $f(x-3)=f(3-x)\ \forall x\iff f(z)=f(-z)\ \forall z,$
which means symmetry about 0, not about 3.

Hence statement (B) is FALSE.

Since X is continuous, $P(X=3)=0$
$\Rightarrow P(X-3=3-X)=P(X=3)=0,$

so (A) is TRUE.

$\boxed{\text{The FALSE statement is (B).}}$

Q5 Text Solution:

Solution:

Let $U=X_1X_2,$
where X_1 and X_2 are independent with $f(x)=\dfrac{1}{\pi(1+x^2)}.$

By the standard formula for the product of two independent random var

$$f_U(u) = \int_{-\infty}^{\infty} f(x)\, f\left(\frac{u}{x}\right) \frac{1}{|x|}\, dx.$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)} \cdot \frac{1}{\pi\left(1+\left(\frac{u}{x}\right)^2\right)} \cdot \frac{1}{|x|}\, dx.$$

$$= \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{|x|}{(1+x^2)(x^2+u^2)}\, dx = \frac{2}{\pi^2} \int_0^{\infty} \frac{x}{(1+x^2)(x^2+u^2)}\, dx.$$

Put $t = x^2 \Rightarrow x\,dx = \frac{1}{2}dt \Rightarrow f_U(u) = \frac{1}{\pi^2} \int_0^{\infty} \frac{1}{(1+t)(t+u^2)}\, dt.$

Using partial fractions, $\frac{1}{(1+t)(t+u^2)}$

$$= \frac{1}{u^2-1}\left(\frac{1}{1+t} - \frac{1}{t+u^2}\right).$$

Hence $\quad f_U(u) = \frac{1}{\pi^2(u^2-1)} \int_0^{\infty} \left(\frac{1}{1+t} - \frac{1}{t+u^2}\right) dt.$

Evaluating the integral gives $2\log|u|$.

$$\therefore \quad f_U(u) = \frac{2\log|u|}{\pi^2(u^2-1)}, \qquad u \in \mathbb{R}.$$

Hence the correct option is (B).

Q6 Text Solution:

Solution:

We are given $X_1, X_2, \ldots$ i.i.d.\ $N(0,1)$,

$$Z_n = \frac{\sum_{i=1}^{n} \frac{X_{2i-1}}{X_{2i}}}{\sum_{i=1}^{2n} X_i^2}.$$

Write $U_n = \sum_{i=1}^{n} \frac{X_{2i-1}}{X_{2i}}, \qquad V_n = \sum_{i=1}^{2n} X_i^2,$

so that $Z_n = \frac{U_n}{V_n}$.

Result 1:

If X, Y are independent $N(0,1)$, then $X/Y \sim \text{Cauchy}(0,1)$.

Here each pair (X_{2i-1}, X_{2i}) is independent and $N(0,1)$,

$$\Rightarrow Y_i = \frac{X_{2i-1}}{X_{2i}} \sim \text{Cauchy}(0,1), \quad i = 1, \ldots, n,$$

and $Y_1, \ldots, Y_n$ are i.i.d.

Result 2 (Additive Property):

If $Y_1, \ldots, Y_n$ are i.i.d.\ Cauchy$(0,1)$, then $Y_1 + \cdots + Y_n \sim \text{Cauchy}(0,n)$,

and hence $\frac{Y_1 + \cdots + Y_n}{n} \sim \text{Cauchy}(0,1)$.

Since $U_n = \sum_{i=1}^{n} Y_i$,

$$\frac{U_n}{n} \sim \text{Cauchy}(0,1) \quad \text{for every } n,$$

$$\Rightarrow \frac{U_n}{n} \xrightarrow{d} \text{Cauchy}(0,1).$$

Result 3 (WLLN):

$$V_n = \sum_{i=1}^{2n} X_i^2 \sim \chi^2_{2n}, \quad E(X_1^2) = 1.$$

$$\frac{1}{2n} V_n = \frac{1}{2n} \sum_{i=1}^{2n} X_i^2 \xrightarrow{a.s.} E(X_1^2) = 1,$$

$$\Rightarrow \frac{V_n}{n} = 2 \cdot \frac{V_n}{2n} \xrightarrow{p} 2.$$

Rewrite Z_n and apply Slutsky's theorem:

$$Z_n = \frac{U_n}{V_n} = \frac{U_n/n}{V_n/n}.$$

$$\frac{U_n}{n} \xrightarrow{d} \text{Cauchy}(0,1), \qquad \frac{V_n}{n} \xrightarrow{p} 2.$$

Result 4 (Slutsky):

If $A_n \xrightarrow{d} A,\ B_n \xrightarrow{p} b \neq 0,$ then $A_n/B_n \xrightarrow{d} A/b.$

Hence $\quad Z_n = \frac{U_n/n}{V_n/n} \xrightarrow{d} \frac{\text{Cauchy}(0,1)}{2}.$

Result 5 (scaling of Cauchy):

If $C \sim \text{Cauchy}(0,\gamma)$ and $a \neq 0$, then $aC \sim \text{Cauchy}(0, |a|\gamma)$.

Thus if $C \sim \text{Cauchy}(0,1)$, then $\frac{1}{2}C \sim \text{Cauchy}\left(0, \frac{1}{2}\right)$.

$$\Rightarrow Z_n \xrightarrow{d} \text{Cauchy}\left(0, \frac{1}{2}\right).$$

The limiting distribution of Z_n is Cauchy $\left(0, \frac{1}{2}\right)$, so the correct option is (D).

Q7 Text Solution:

Solution:

Given:
Consider the probability space $(\Omega, \mathcal{F}, P) = ((0,1),\ \mathcal{B}(0,1),\ P_U)$, where $P_U((0,x]) = x,\ x \in (0,1)$.

Define $X(\omega) = \log \dfrac{1}{1-\omega}, \qquad \omega \in (0,1)$.

$$X(\omega) = \log \frac{1}{1-\omega} = -\log(1-\omega),\quad 0 < \omega < 1.$$

Proof that X is exponential:

$$P(X \le x) = P(-\log(1-\omega) \le x) = P(1-\omega \le e^{-x}) = P(\omega \ge 1 - e^{-x}).$$

Since $\omega \sim \text{Uniform}(0,1)$,

$$P(\omega \ge 1 - e^{-x}) = e^{-x}, \qquad x \ge 0.$$

$$\Rightarrow\ P(X \le x) = 1 - e^{-x}, \qquad x \ge 0,$$

which is the CDF of the exponential distribution with parameter 1.

Hence $X \sim \text{Exponential}(1)$.

Result: $E(e^{-tX}) = \dfrac{1}{1+t}, \qquad t > -1.$

Step 1: $E\left(e^{-\frac{3}{4}X}\right) = \dfrac{1}{1+\frac{3}{4}} = \dfrac{4}{7}.$

Step 2: $E\left(e^{-\frac{3}{2}X}\right) = \dfrac{1}{1+\frac{3}{2}} = \dfrac{2}{5}.$

Step 3: $\text{Var}\left(e^{-\frac{3}{4}X}\right) = E\left(e^{-\frac{3}{2}X}\right) - \left(E\left(e^{-\frac{3}{4}X}\right)\right)^2.$

$$= \frac{2}{5} - \frac{16}{49} = \frac{98-80}{245} = \frac{18}{245}.$$

$$\boxed{\text{Var}\left(e^{-\frac{3}{4}X}\right) = \frac{18}{245}.}$$

Correct option is (C).

Q8 Text Solution:

Solution:

$$X \sim \text{Uniform}(2,12), \quad f_X(x) = \frac{1}{12-2} = \frac{1}{10},\ 2 \le x \le 12.$$

$$Z = \begin{cases} X-4, & X \le 8, \\ X+2, & X > 8. \end{cases}$$

General Formula:

$$\text{If } Z = \begin{cases} Z_1(X), & X \in A, \\ Z_2(X), & X \in A^c, \end{cases}$$

then $E(Z) = E(Z_1(X)\mathbf{1}_A) + E(Z_2(X)\mathbf{1}_{A^c})$.

In the continuous case, $E(Z) = \displaystyle\int_A Z_1(x) f_X(x)\,dx + \int_{A^c} Z_2(x) f_X(x)\,dx.$

Write $E(Z)$ as an integral:

$$E(Z) = E\big((X-4)\mathbf{1}_{\{X\le 8\}} + (X+2)\mathbf{1}_{\{X>8\}}\big)$$

$$= \int_2^8 (x-4)\frac{1}{10}\,dx + \int_8^{12} (x+2)\frac{1}{10}\,dx.$$

First integral:

$$\int_2^8 (x-4)\frac{1}{10}\,dx = \frac{1}{10}\int_2^8 (x-4)\,dx = \frac{1}{10}\left[\frac{x^2}{2} - 4x\right]_2^8.$$

$$\left[\frac{x^2}{2} - 4x\right]_2^8 = \left(\frac{8^2}{2} - 4\cdot 8\right) - \left(\frac{2^2}{2} - 4\cdot 2\right) = (32-32) - (2-8) = 0 - (-6) = 6.$$

$$\Rightarrow \int_2^8 (x-4)\frac{1}{10}\,dx = \frac{6}{10} = 0.6.$$

Second integral:

$$\int_8^{12} (x+2)\frac{1}{10}\,dx = \frac{1}{10}\int_8^{12} (x+2)\,dx = \frac{1}{10}\left[\frac{x^2}{2} + 2x\right]_8^{12}.$$

$$\left[\frac{x^2}{2} + 2x\right]_8^{12} = \left(\frac{12^2}{2} + 2\cdot 12\right) - \left(\frac{8^2}{2} + 2\cdot 8\right) = (72+24) - (32+16) = 96 - 48 = 48.$$

$$\Rightarrow \int_8^{12} (x+2)\frac{1}{10}\,dx = \frac{48}{10} = 4.8.$$

Add both parts:

$$E(Z) = 0.6 + 4.8 = 5.4.$$

$$\boxed{E(Z) = 5.4.}$$

Correct option: (B)

Q9 Text Solution:

Solution.

$$P\left(\frac{X}{2} \leq \theta \leq 2X\right) = P\left(X \geq \frac{\theta}{2}\right) = \int_{\theta/2}^{\theta} \frac{3x^2}{\theta^3}\, dx.$$

$$= \frac{3}{\theta^3}\left[\frac{x^3}{3}\right]_{\theta/2}^{\theta} = \frac{1}{\theta^3}\left(\theta^3 - \left(\frac{\theta}{2}\right)^3\right) = 1 - \frac{1}{8} = \frac{7}{8}.$$

$$\therefore \text{ Confidence coefficient } = \frac{7}{8}.$$

Q10 Text Solution:

Solution.

$$L\left(\theta\right) \propto \left(\frac{\theta}{3}\right)^{40}\left(\frac{2(1-\theta)}{3}\right)^{40}\left(\frac{\theta+1}{3}\right)^{40} = c\,\theta^{40}\,(1-\theta)^{40}(1+\theta)^{40}.$$

$$\ell(\theta) = \ln L(\theta) = \ln c + 40\ln\theta + 40\ln(1-\theta) + 40\ln(1+\theta).$$

$$\frac{d\ell(\theta)}{d\theta} = \frac{40}{\theta} - \frac{40}{1-\theta} + \frac{40}{1+\theta} = 0$$

$$\Rightarrow\ 1 - 3\theta^2 \Rightarrow \hat{\theta} = \frac{1}{\sqrt{3}}.$$

$$\therefore\ \text{The MLE of } \theta \text{ is } \boxed{\frac{1}{\sqrt{3}}}.$$

Hence Option A is TRUE.

Q11 Text Solution:

Given $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x^3 + x + a$$
$$\Rightarrow f'(x) = 3x^2 + 1 > 0 \forall x$$

$\Rightarrow f$ is an increasing function, hence it cuts the x -axis only once.

Thus, it has exactly one real root of for all $\boldsymbol{a}$.

So option (b) is correct answer.

Q12 Text Solution:

Let $a_n = \frac{\pi}{2} + 2n\pi$ and $b_n = \frac{3\pi}{2} + 2n\pi$

Then $a_n \to \infty,\ b_n \to \infty$

but $\lim\limits_{n\to\infty} f(a_n) = \lim\limits_{n\to\infty} \left(\frac{\pi}{2} + 2n\pi\right)^2 \to \infty$

and $\lim\limits_{n\to\infty} f(b_n) = \lim\limits_{n\to\infty} \left(\frac{3\pi}{2} + 2n\pi\right)^{1-1} = 1$

Hence, by sequential criteria limit does not exists.

Q13 Text Solution:

$$A = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 1 & -2 & 0 \\ \hline 0 & 0 & -3 \end{array}\right]$$

Eigenvalues of A are 1, $-2, -3$

$\Rightarrow\ C_A(x) = (x-1)(x+2)(x+3)$

$= (x-1)\left(x^2 + 5x + 6\right)$

$= x^3 + 5x^2 + 6x - x^2 - 5x - 6$

$C_A(x) = x^3 + 4x^2 + x - 6$

Using, Cayley's Hamilton theorem

$\Rightarrow A_3 + 4A^2 + A - 6I = 0$

$\Rightarrow 6I = A^3 + 4A^2 + A$

Apply A^{-1} on both sides, we get

$\Rightarrow 6A^{-1} = A^2 + 4A + I$

$\Rightarrow 6A^{-1} = aA^2 + bA + cI$ for $a, b, c \in \mathbb{R}$

Compare with above equation (i), we get

a = 1, b = 4, c = 1

So, (a, b, c) = (1, 4, 1)

Q14 Text Solution:

Note that $k\alpha - 1 \leq [k\alpha] \leq k\alpha\ \ \forall k \in \mathbb{N}$.

$\Rightarrow \frac{\alpha-1+2\alpha-1+\ldots+n\alpha-1}{n^2} \leq \frac{[\alpha]+[2\alpha]+\ldots+[n\alpha]}{n^2}$

$\leq \frac{\alpha+2\alpha+\ldots+n\alpha}{n^2}$

$\Rightarrow \frac{\frac{\alpha n(n+1)}{2} - n}{n^2} \leq x_n \leq \frac{\alpha n(n+1)}{2n^2}$

Clearly, $\lim\limits_{n\to\infty} \frac{\alpha n(n+1)}{2n^2} = \frac{\alpha}{2}$

$= \lim\limits_{n\to\infty} \frac{\frac{\alpha n(n+1)}{2} - n}{n^2}$

So, by squeeze theorem

$\lim\limits_{n\to\infty} x_n = \frac{\alpha}{2}$

Q15 Text Solution:

Consider $\frac{a_{n+1}}{a_n} = \frac{a_n}{\frac{1+a_n^2}{a_n}}$

$= \frac{1}{1+a_n^2} \leq 1\ \ \left[\text{since } a_n^2 \geq 0\right]$

$\Rightarrow a_{n+1} \leq a_n; \forall n \in \mathbb{N}$

$\Rightarrow <a_n>$ is monotonically decreasing

Now, for bounds using PMI

For $n = 1$

$a_2 = \frac{a_1}{1+a_1^2} > 0$

Let $a_k = \frac{a_{k-1}}{1+a_{k-1}^2} > 0$

Then $a_{k+1} = \frac{a_k}{1+a_k^2} > 0$

$\Rightarrow\ \ a_n > 0\ \ \forall n \in \mathbb{N}$

Q16 Text Solution:

Case (i) : Let $x \in \mathbb{Q}$ then by choosing n to be sufficiently large, $n!x$ can be made an integer, then $\sin(n!\pi x) = 0$.

$\Rightarrow f\left(x\right) = \lim\limits_{t\to 0} \frac{0}{0+t^2} = 0$

Case (ii) let $x \in \mathbb{Q}^c$, then $0 < \sin^2\left(n!\pi x\right) < 1$

$$\Rightarrow f\left(x\right) = \lim_{n\to\infty}\left[\lim_{t\to 0} \frac{1}{1+\frac{t^2}{\sin^2\left(n!\pi x\right)}}\right] = 1$$

$$\Rightarrow f\left(x\right) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{Q}^c \end{cases}$$

Hence, f is nowhere continuous on $\mathbb{R}$.

Q17 Text Solution:

Solution.

$$\text{Let } Y_i = |X_i|, \qquad Z_i = [Y_i] = [|X_i|],$$

$$S_n = \sum_{i=1}^{n} Z_i.$$

Since the density of X_i is symmetric, Y_i has $f_Y(y) = e^{-y},\ y > 0, \Rightarrow Y_i \sim \text{Exponential}(1)$.

Distribution of $Z_1 = [Y_1]$:

$$P(Z_1 = k) = P(k \le Y_1 < k+1)$$

$$= \int_k^{k+1} e^{-y}\,dy$$

$$= (1-e^{-1})\,e^{-k}, \qquad k = 0, 1, 2, \dots$$

Result:

For the geometric distribution on $\{0, 1, 2, \dots\}$,

$$E(Z_1) = \frac{1-p}{p}, \qquad \mathrm{Var}(Z_1) = \frac{1-p}{p^2},$$

with $p = 1 - e^{-1}$.

$$E(Z_1) = \frac{1}{e-1}, \qquad \mathrm{Var}(Z_1) = \frac{e}{(e-1)^2}.$$

Using CLT:

$$\frac{S_n - a_n}{b_n} \xrightarrow{d} N(0,1)$$

for $a_n = nE(Z_1) = \dfrac{n}{e-1}$,

$$b_n = \sqrt{n\,\mathrm{Var}(Z_1)} = \frac{\sqrt{e\,n}}{e-1}.$$

Then $\dfrac{b_n}{a_n} = \sqrt{\dfrac{e}{n}}, \qquad 1 - \dfrac{b_n}{a_n} = 1 - \sqrt{\dfrac{e}{n}}$

.

Result: $\log(1-x) \sim -x$ as $x \to 0$.

$$\log\left(1 - \frac{b_n}{a_n}\right) = \log\left(1 - \sqrt{\frac{e}{n}}\right) \sim -\sqrt{\frac{e}{n}}.$$

$$\frac{1}{2}\sqrt{n}\,\log\left(1 - \frac{b_n}{a_n}\right) \sim \frac{1}{2}\sqrt{n}\left(-\sqrt{\frac{e}{n}}\right) = -\frac{\sqrt{e}}{2}.$$

$$\boxed{\lim_{n\to\infty} \frac{1}{2}\sqrt{n}\,\log\left(1 - \frac{b_n}{a_n}\right) = -\frac{\sqrt{e}}{2}.}$$

Correct option: (B) $-\dfrac{\sqrt{e}}{2}$.

Q18 Text Solution:

Solution.

The density is $f_\theta(x) = \begin{cases} \dfrac{\theta}{x^2}, & x \ge \theta, \\ 0, & \text{otherwise}, \end{cases}$

$\theta > 0.$

For a random sample $X_1, \dots, X_n$, let $x_{(1)} = \min(x_1, \dots, x_n)$.

The joint density under θ is $f_\theta(x_1, \dots, x_n)$

$$= \theta^n \left(\prod_{i=1}^n x_i^{-2}\right) I\{x_{(1)} \ge \theta\}.$$

Hence the likelihood is $L(\theta \mid x)$

$$= \theta^n I\{\theta \le x_{(1)}\}\left(\prod_{i=1}^n x_i^{-2}\right).$$

For testing $H_0 : \theta = 1$ against $H_A : \theta = 2$,

the likelihood ratio is $\Lambda(x) = \dfrac{L(2 \mid x)}{L(1 \mid x)}$

$$= 2^{\,n}\, I\{x_{(1)} \ge 2\}.$$

Since the family has a monotone likelihood ratio in $x_{(1)}$, the MP test rejects for large $x_{(1)}$.

Thus the critical region has the form $\{(x_1, \dots, x_n) : x_{(1)} > c\}$.

To find c, impose the size condition under $H_0 : \theta = 1$.

$$P_1(X_{(1)} > x) = P_1(X_1 > x, \dots, X_n > x) = (P_1(X_1 > x))^n.$$

$$P_1(X_1 > x) = \int_x^\infty \frac{1}{t^2}\,dt = \frac{1}{x}, \qquad x \ge 1.$$

$$\Rightarrow \quad P_1(X_{(1)} > x) = \left(\frac{1}{x}\right)^n, \qquad x \ge 1.$$

The size α condition is $P_1(X_{(1)} > c) = \alpha$.

$$\Rightarrow \quad \left(\frac{1}{c}\right)^n = \alpha \quad \Rightarrow \quad c = \alpha^{-1/n}.$$

Hence the MP size α critical region is $\{(x_1, \dots, x_n) : x_{(1)} > \alpha^{-1/n}\}$.

$$\boxed{\text{Reject } H_0 \text{ if } x_{(1)} > \alpha^{-1/n}.}$$

Q19 Text Solution:

Solution.

On the support we have $0 \le x \le \dfrac{1}{3}$,

$$x^2 \le y \le 5x^2,$$

and since $x \ge 0,\ |x| = x,$ so $f(x,y) = kx$.

$$1 = \iint_{\text{support}} kx\,dy\,dx = \int_0^{1/3}\int_{x^2}^{5x^2} kx\,dy\,dx = \int_0^{1/3} kx(5x^2 - x^2)\,dx = \int_0^{1/3} 4kx^3\,dx.$$

$$\int_0^{1/3} x^3\,dx = \left.\frac{x^4}{4}\right|_0^{1/3} = \frac{1}{4 \cdot 3^4} = \frac{1}{324}.$$

$$\Rightarrow \ 1 = 4k \cdot \frac{1}{324} = \frac{k}{81} \quad \Rightarrow \quad k = 81.$$

Thus $f(x,y) = 81x$ on the support.

The event $2X+3Y>1$ is equivalent to $y > \frac{1-2x}{3}$.

We must solve the intersections of $y = \frac{1-2x}{3}$ with $y=x^2$ and $y=5x^2$.

$$x^2 = \frac{1-2x}{3} \Rightarrow 3x^2+2x-1=0 \Rightarrow$$
$$(3x-1)(x+1)=0 \Rightarrow x=\frac{1}{3} \text{ in } [0,1/3].$$

$$5x^2 = \frac{1-2x}{3} \Rightarrow 15x^2+2x-1=0 \Rightarrow (5x-1)(3x+1)=0 \Rightarrow x = \frac{1}{5} \text{ in } [0,1/3].$$

Hence on $0 \le x \le \frac{1}{5}$, $5x^2 \le \frac{1-2x}{3}$, so there is no contribution to $P(2X+3Y>1)$.

On $\frac{1}{5} < x < \frac{1}{3}$, $x^2 < \frac{1-2x}{3} < 5x^2$, so the event region is $y \in \left(\frac{1-2x}{3}, 5x^2\right)$.

$$P(2X+3Y>1) = \int_{1/5}^{1/3} \int_{(1-2x)/3}^{5x^2} 81x\,dy\,dx.$$

$$= \int_{1/5}^{1/3} 81x\left(5x^2 - \frac{1-2x}{3}\right) dx = \int_{1/5}^{1/3} 81x\left(5x^2+\frac{2x}{3}-\frac{1}{3}\right) dx.$$

$$= 81\int_{1/5}^{1/3}\left(5x^3+\frac{2}{3}x^2-\frac{1}{3}x\right) dx.$$

$$\int\left(5x^3+\frac{2}{3}x^2-\frac{1}{3}x\right) dx = \frac{5}{4}x^4+\frac{2}{9}x^3-\frac{1}{6}x^2.$$

$$\left[\frac{5}{4}x^4+\frac{2}{9}x^3-\frac{1}{6}x^2\right]_{x=1/5}^{x=1/3} = \frac{976}{121500}.$$

$$\Rightarrow P(2X+3Y>1) = 81\cdot\frac{976}{121500} = \frac{244}{375}.$$

$$\boxed{P(2X+3Y>1)=\frac{244}{375}.}$$

Q20 Text Solution:

Solution.

$X_1, X_2 \sim \text{Bernoulli}\left(\frac{1}{2}\right)$ independent, and $M = \min(X_1, X_2)$.

Since $X_i \in \{0,1\}$, $M = \min(X_1,X_2) = \begin{cases} 1, & \text{if } X_1=1 \text{ and } X_2=1, \\ 0, & \text{otherwise.} \end{cases}$

$$\Rightarrow M = X_1X_2.$$

$$E(X_2) = \frac{1}{2},$$

$$E(M) = P(X_1=1, X_2=1) = \frac{1}{4}.$$

$$E(X_2M) = E(X_2^2X_1) = E(X_2X_1) = \frac{1}{4}.$$

$$\text{Cov}(X_2,M) = E(X_2M) - E(X_2)E(M) = \frac{1}{4} - \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}.$$

$$\text{Var}(X_2) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}, \qquad \text{Var}(M) = \frac{1}{4}\cdot\frac{3}{4} = \frac{3}{16}.$$

$$\text{Corr}(X_2,M) = \frac{\text{Cov}(X_2,M)}{\sqrt{\text{Var}(X_2)}\sqrt{\text{Var}(M)}} = \frac{\frac{1}{8}}{\sqrt{\frac{1}{4}}\sqrt{\frac{3}{16}}} = \frac{1}{\sqrt{3}}.$$

$$\boxed{\frac{1}{\sqrt{3}}}$$

Q21 Text Solution:

Solution.

Let N denote the total number of trains in one hour and F the number c

$$N \sim \text{Poisson}(80), \qquad P(\text{freight}) = 0.2, \quad P(\text{passenger}) = 0.8.$$

By thinning of the Poisson process, $F \sim \text{Poisson}(80\times 0.2) = \text{Poisson}(16)$,

and $P \sim \text{Poisson}(80\times 0.8) = \text{Poisson}(64)$, with $F \perp P$.

Also $N = F + P$.

$$E[N \mid F=16] = E[F \mid F=16] + E[P \mid F=16].$$

$$E[F \mid F=16] = 16, \qquad E[P \mid F=16] = E[P] = 64.$$

$$E[N \mid F=16] = 16+64 = 80.$$

$$\boxed{E[N \mid F=16] = 80.}$$

Let $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda > 0$.

Each arrival is independently classified as type I with probability p and as type II with probability $1-p$.

Let $N_1(t)$ and $N_2(t)$ denote the corresponding counting processes of type I and type II arrivals.

Then $N_1(t) \sim \text{Poisson}(\lambda p t)$, $N_2(t) \sim \text{Poisson}(\lambda(1-p)t)$,

and $N_1(t) \perp N_2(t), \qquad t \geq 0.$

Q22 Text Solution:

Solution.

We have $U = X + Y, \quad V = 4X - 2Z, \quad W = 3Y + Z,$ where X, Y, Z are independent $N(0, 1)$ random variables.

All variables are jointly Gaussian, so we can use

$$\rho_{VW|U} = \frac{\rho_{VW} - \rho_{VU}\rho_{WU}}{\sqrt{(1-\rho_{VU}^2)(1-\rho_{WU}^2)}}.$$

First compute variances.

$$\text{Var}(U) = \text{Var}(X+Y) = 1 + 1 = 2.$$

$$\text{Var}(V) = \text{Var}(4X - 2Z) = 4^2 \cdot 1 + (-2)^2 \cdot 1 = 16 + 4 = 20.$$

$$\text{Var}(W) = \text{Var}(3Y + Z) = 3^2 \cdot 1 + 1^2 \cdot 1 = 9 + 1 = 10.$$

Next compute covariances using independence of X, Y, Z.

$$\text{Cov}(V, W) = \text{Cov}(4X - 2Z,\ 3Y + Z) = 4 \cdot 3\ \text{Cov}(X, Y) + 4 \cdot 1\ \text{Cov}(X, Z) - 2 \cdot 3\ \text{Cov}(Z, Y) - 2 \cdot 1\ \text{Var}(Z) = -2.$$

$$\text{Cov}(V, U) = \text{Cov}(4X - 2Z,\ X + Y) = 4\ \text{Var}(X) = 4.$$

$$\text{Cov}(W, U) = \text{Cov}(3Y + Z,\ X + Y) = 3\ \text{Var}(Y) = 3.$$

Now compute the ordinary correlations.

$$\rho_{VW} = \frac{\text{Cov}(V, W)}{\sqrt{\text{Var}(V)\,\text{Var}(W)}} = \frac{-2}{\sqrt{20 \cdot 10}} = -\frac{1}{5\sqrt{2}}.$$

$$\rho_{VU} = \frac{\text{Cov}(V, U)}{\sqrt{\text{Var}(V)\,\text{Var}(U)}} = \frac{4}{\sqrt{20 \cdot 2}} = \frac{2}{\sqrt{10}}.$$

$$\rho_{WU} = \frac{\text{Cov}(W, U)}{\sqrt{\text{Var}(W)\,\text{Var}(U)}} = \frac{3}{\sqrt{10 \cdot 2}} = \frac{3}{2\sqrt{5}}.$$

Compute the numerator of the partial correlation:

$$\rho_{VW} - \rho_{VU}\rho_{WU} = -\frac{1}{5\sqrt{2}} - \left(\frac{2}{\sqrt{10}}\right)\left(\frac{3}{2\sqrt{5}}\right) = -\frac{1}{5\sqrt{2}} - \frac{3}{5\sqrt{2}} = -\frac{4}{5\sqrt{2}}.$$

Compute the denominator:

$$1 - \rho_{VU}^2 = 1 - \left(\frac{2}{\sqrt{10}}\right)^2 = 1 - \frac{4}{10} = \frac{3}{5},$$

$$1 - \rho_{WU}^2 = 1 - \left(\frac{3}{2\sqrt{5}}\right)^2 = 1 - \frac{9}{20} = \frac{11}{20}.$$

$$\sqrt{(1-\rho_{VU}^2)(1-\rho_{WU}^2)} = \sqrt{\frac{3}{5} \cdot \frac{11}{20}} = \sqrt{\frac{33}{100}} = \frac{\sqrt{33}}{10}.$$

Therefore

$$\rho_{VW|U} = \frac{-\frac{4}{5\sqrt{2}}}{\frac{\sqrt{33}}{10}} = -\frac{4}{5\sqrt{2}} \cdot \frac{10}{\sqrt{33}} = -\frac{40}{5\sqrt{66}} = -\frac{8}{\sqrt{66}}.$$

$$\boxed{\rho_{VW|U} = -\frac{8}{\sqrt{66}}.}$$

Hence the correct option is C.

Q23 Text Solution:

Let X_1 , X_2 , $\cdots$, X_n
be random sample of size n (≥ 2)
from a distribution having
probability density function

$$f(x,\,\theta) = \begin{cases} \frac{1}{3}e^{-\frac{(x-\theta)}{3}}, & x > \theta \\ 0, & \text{otherwise} \end{cases}$$

where $\theta \in \mathbb{R}$. Let $X_{(1)}$
$= \min(X_1,\ X_2,\ \cdots,\ X_n)$.
Now , $X_{(1)}$ is Complete sufficient for θ.
Since
$E\left(X_1 + e^{|X_1-X_3|} - e^{|X_2-X_4|}\right) = E(X_1)$
$\Rightarrow \theta + 3$
since X_i are i.i.d. random variables.
$E\left(e^{|X_1-X_3|}\right) = E\left(e^{|X_2-X_4|}\right)$
By Rao Blackwell theorem,
$\mathbb{E}\left(X_1 + e^{|X_1-X_3|} - e^{|X_1-X_3|}|X_{(1)}\right)$
$= h\left(X_{(1)}\right)$
$E\left(X_{(1)}\right) = \theta + \frac{3}{n}$
$E\left(X_{(1)} - \frac{3}{n}\right) = \theta$
$h\left(X_{(1)}\right) = X_{(1)} - \frac{3}{n} + 3$
$= X_{(1)} + 3\frac{(n-1)}{n}$
Hence Correct option is C.

Q24 Text Solution:

The random variable X takes the values $\{-1\ ,\ 0,\ 1\}$according to the distribution $\{F_p : 0 < p < \frac{1}{3}\}$ such that X

$$= \begin{cases} -1 & \text{with probability} \quad p, \\ 0 & \text{with probability} \quad 2p, \\ 1 & \text{with probability} \quad 1-3p. \end{cases}$$

Let X_1 and X_2
be random sample from this distribution.
Define $T = |X_1 + X_2|$

Completeness of X :

$E(g(X)) = 0$
$\Rightarrow g(-1) \times p + g(0) \times 2p + g(1)$
$\times (1-3p) = 0$, for all $p \in \left(0, \frac{1}{3}\right)$
$\Rightarrow p(g(-1) + 2g(0) - 3g(1)) + g(1) = 0$
, for all $p \in \left(0, \frac{1}{3}\right)$
so, $g(1) = 0$ *and* $g(-1) + 2g(0) = 0$,
for all $p \in \left(0, \frac{1}{3}\right)$
now,
consider the function g

$$g(x) = \begin{cases} 1, & x = 0 \\ -2, & x = -1 \\ 0, & x = 1. \end{cases}$$

Then $\mathbb{E}(g(X)) = 0$, for all $p \in \left(0, \frac{1}{3}\right)$
But $P(g(X) = 0) \neq 1$.
So X is not complete.

Completeness of T :

Distribution of T :

$P(T=0) = P(X_1 = 0,\ X_2 = 0)$
$+ P(X_1 = 1,\ X_2 = -1)$
$+ P(X_1 = -1,\ X_2 = 1)$
$= 4p^2 + 2p(1-3p)$
Similarly,
$P(T=1) = 4p^2 + 4p(1-3p)$
$P(T=2) = p^2 + (1-3p)^2$
Now,
$E(h(T)) = 0$
$\Rightarrow h(0) \times \left(4p^2 + 2p(1-3p)\right) + h(1)$
$\times \left(4p^2 + 4p(1-3p)\right) + h(2)$
$\times \left(p^2 + (1-3p)\right)^2 = 0,$ for all p
$\in \left(0, \frac{1}{3}\right)$
$\Rightarrow 2p^2\{-h(0) - 4h(1) + h(2)\}$
$+ 2p\{h(0) + 2h(1) - 3h(2)\} + h(2)$
$= 0,$ for all $p \in \left(0, \frac{1}{3}\right)$
$\Rightarrow h(2) = 0$, $-h(0) = h(1)$ *and* $2h(1)$
$= -h(0),$ for all $p \in \left(0, \frac{1}{3}\right)$
$\Rightarrow h(0) = h(1) = h(2) = 0,$ for all p
$\in \left(0, \frac{1}{3}\right)$
Then $\mathbb{E}(h(T)) = 0$, for all $p \in \left(0, \frac{1}{3}\right)$.
So , T is complete.
Hence D is correct option.

Q25 Text Solution:

$$X \sim N(0,9), \quad P(Y=1) = P(Y=-1) = \tfrac{1}{2}, \quad X \perp Y,$$

$$Z = XY + \frac{X}{Y}.$$

For $Y=1: \quad Z = X\cdot 1 + \frac{X}{1} = 2X.$

For $Y=-1: \quad Z = X\cdot(-1) + \dfrac{X}{-1} = -X - X = -2X.$

$$\therefore\ Z = \begin{cases} 2X, & \text{with prob. } \frac{1}{2}, \\ -2X, & \text{with prob. } \frac{1}{2}. \end{cases}$$

$$2X \sim N(0,36), \quad -2X \sim N(0,36).$$

$$F_Z(z) = P(Z \le z) = P(Z \le z \mid Y=1)P(Y=1) + P(Z \le z \mid Y=-1)P(Y=-1).$$

$$F_Z(z) = \tfrac{1}{2}P(2X \le z) + \tfrac{1}{2}P(-2X \le z).$$

$$P(2X \le z) = P\left(X \le \frac{z}{2}\right) = P\left(\frac{X}{3} \le \frac{z}{6}\right) = \Phi\left(\frac{z}{6}\right).$$

$$P(-2X \le z) = P\left(X \ge -\frac{z}{2}\right) = 1 - \Phi\left(-\frac{z}{6}\right) = \Phi\left(\frac{z}{6}\right).$$

$$\therefore\ F_Z(z) = \tfrac{1}{2}\Phi\left(\frac{z}{6}\right) + \tfrac{1}{2}\Phi\left(\frac{z}{6}\right) = \Phi\left(\frac{z}{6}\right),$$

$$\boxed{Z \sim N(0,36) \text{ and } F_Z(z) = \Phi\left(\frac{z}{6}\right)}.$$

Now,

$$Z^3 - 2Z^2 - Z + 2 = (Z-2)(Z-1)(Z+1).$$

$$(Z-2)(Z-1)(Z+1) < 0 \iff Z \in (-\infty,-1) \cup (1,2).$$

$$P(Z^3 - 2Z^2 - Z + 2 < 0) = P(Z < -1) + P(1 < Z < 2).$$

$$Z \sim N(0,6^2) \ \Rightarrow\ \frac{Z}{6} \sim N(0,1).$$

$$P(Z < -1) = \Phi\left(-\frac{1}{6}\right),$$

$$P(1 < Z < 2) = \Phi\left(\frac{2}{6}\right) - \Phi\left(\frac{1}{6}\right).$$

$$\boxed{P(Z^3 - 2Z^2 - Z + 2 < 0) = \Phi\left(-\tfrac{1}{6}\right) + \Phi\left(\tfrac{1}{3}\right) - \Phi\left(\tfrac{1}{6}\right)}$$

Hence D is correct option.

Q26 Text Solution:

We have $X_i \sim N(\theta, 5)$

For the normal family $N(\theta, 5)$ (variance known), $\bar{X}$ is complete and sufficient for θ.

If an estimator has the form $T(X) = h(\bar{X})$, where h is one–to–one,

Consider $h(x) = x + \exp(2(x-1)),\ x \in \mathbb{R}.$

$$h'(x) = 1 + 2\exp(2(x-1)).$$

Since $\exp(\cdot) > 0$, we have $h'(x) = 1 + 2\exp(2(x-1)) > 1 > 0$ for all $x \in \mathbb{R}$.

Therefore $h'(x) > 0$ everywhere, so h is strictly increasing.

Hence h is one--one (injective).

then $T(X)$ and $\bar{X}$ contain the same information about θ.

Hence $T(X)$ is also a (one–to–one) function of the complete sufficient statistic $\bar{X}$.

Therefore any unbiased estimator of this form is the UMVUE (Lehmann

Given $n = 5,\ \sigma^2 = 5.$

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) = N(\mu, 1).$$

(a) UMVUE of $e^{2\mu}$.

$$E(e^{2\bar{X}}) = \exp\left(2\mu + \frac{2\sigma^2}{n}\right) = \exp(2\mu + 2).$$

Let $\hat{g}(\bar{X}) = c\,e^{2\bar{X}}$.

$$E(\hat{g}) = c\,e^{2\mu+2} = e^{2\mu} \Rightarrow c = e^{-2}.$$

$$\boxed{\hat{g}(\bar{X}) = e^{-2}e^{2\bar{X}}}.$$

(b) CRLB for estimating $e^{2\mu}$.

$$g(\mu) = e^{2\mu} \quad\Rightarrow\quad g'(\mu) = 2e^{2\mu}.$$

$$I_5(\mu) = \frac{5}{\sigma^2} = \frac{5}{5} = 1.$$

$$\text{CRLB} = \frac{(g'(\mu))^2}{I_5(\mu)} = \frac{(2e^{2\mu})^2}{1} = 4e^{4\mu}.$$

$$\boxed{\text{CRLB} = 4e^{4\mu}}.$$

(c) Variance of the UMVUE.

$$\hat{g} = e^{-2}e^{2\bar{X}}.$$

$$\hat{g}^2 = e^{-4}e^{4\bar{X}}.$$

$$E(e^{4\bar{X}}) = \exp\left(4\mu + \frac{8\sigma^2}{n}\right) = \exp(4\mu + 8)$$

$$E(\hat{g}^2) = e^{-4}e^{4\mu+8} = e^{4\mu+4}.$$

$$\text{Var}(\hat{g}) = E(\hat{g}^2) - (E\hat{g})^2 = e^{4\mu+4} - e^{4\mu} = e^{4\mu}(e^4 - 1).$$

$$\boxed{\text{Var}(\hat{g}) = e^{4\mu}(e^4 - 1)}.$$

(d) Comparison.

Since the exponential function is convex, by the tangent line inequality we have

$$e^x \geq 1 + x \quad \text{for all } x \in \mathbb{R},$$

with strict inequality $e^x > 1 + x$ for $x \neq 0$.

$$e^4 - 1 > 4 \quad \Rightarrow \quad e^{4\mu}(e^4 - 1) > 4e^{4\mu} = \text{CRLB}.$$

Hence the CRLB is not attained.

$$\boxed{\text{UMVUE } = e^{-2}e^{2\bar{X}}, \quad \text{CRLB} = 4e^{4\mu}, \quad \text{Var}(\hat{g}) = e^{4\mu}(e^4 - 1).}$$

Hence D is correct.

Q27 Text Solution:

Solution .

$$X_n = Y_n \bmod 2026 = (Z_1 + Z_2 + \cdots + Z_n) \bmod 2026.$$

Each Z_i is uniform on $\{1, 2, \ldots, 2025\}$.

Modulo 2026, $\{1, 2, \ldots, 2025\}$ are exactly the nonzero residues $1, \ldots, 2025$.

Thus the increment Z_i mod 2026 takes each nonzero residue with probability $\frac{1}{2025}$.

The process (X_n) is a Markov chain on $\{0, 1, \ldots, 2025\}$.

From any state r, $X_{n+1} = (r + Z_{n+1}) \bmod 2026$ can move to many states.

Because the steps cover all nonzero residues, repeated sums generate every residue class mod 2026.

Hence the chain is irreducible.

There exist cycles of different lengths, so the gcd of cycle lengths is 1; thus the chain is aperiodic.

For a finite irreducible aperiodic Markov chain on the group $\mathbb{Z}_{2026}$, with symmetric step support covering all residues, the unique stationary

$$\therefore \; \pi(r) = P_{\text{stat}}(X = r) = \frac{1}{2026}, \quad r = 0, 1, \ldots, 2025.$$

By convergence to stationarity, $P(X_n = r) \to \pi(r)$ as $n \to \infty$.

$$\Rightarrow \lim_{n\to\infty} P(X_n = 2025) = \frac{1}{2026}.$$

$$\boxed{\lim_{n\to\infty} P(X_n = 2025) = \frac{1}{2026}.}$$

Thus, option C is correct.

Q28 Text Solution:

Model: $Y_i = \beta X_i + \varepsilon_i, \quad E(\varepsilon_i) = 0, \;\; \text{Var}(\varepsilon_i) = \sigma^2.$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n X_iY_i}{\sum_{i=1}^n X_i^2}, \qquad \hat{\beta}_2 = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i}.$$

Unbiasedness of $\hat{\beta}_1$:

$$Y_i = \beta X_i + \varepsilon_i \Rightarrow \sum_{i=1}^n X_iY_i = \sum_{i=1}^n X_i(\beta X_i + \varepsilon_i) = \beta\sum_{i=1}^n X_i^2 + \sum_{i=1}^n X_i\varepsilon_i.$$

$$\hat{\beta}_1 = \frac{\beta\sum_{i=1}^n X_i^2 + \sum_{i=1}^n X_i\varepsilon_i}{\sum_{i=1}^n X_i^2} = \beta + \frac{\sum_{i=1}^n X_i\varepsilon_i}{\sum_{i=1}^n X_i^2}.$$

$$E(\hat{\beta}_1) = \beta + \frac{E\left(\sum_{i=1}^n X_i\varepsilon_i\right)}{\sum_{i=1}^n X_i^2} = \beta + \frac{\sum_{i=1}^n X_iE(\varepsilon_i)}{\sum_{i=1}^n X_i^2} = \beta.$$

Thus $\hat{\beta}_1$ is unbiased for β.

Unbiasedness of $\hat{\beta}_2$:

$$\sum_{i=1}^n Y_i = \sum_{i=1}^n(\beta X_i + \varepsilon_i) = \beta\sum_{i=1}^n X_i + \sum_{i=1}^n \varepsilon_i.$$

$$\hat{\beta}_2 = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i} = \beta + \frac{\sum_{i=1}^n \varepsilon_i}{\sum_{i=1}^n X_i}.$$

$$E(\hat{\beta}_2) = \beta + \frac{E\left(\sum_{i=1}^n \varepsilon_i\right)}{\sum_{i=1}^n X_i} = \beta.$$

Hence $\hat{\beta}_2$ is also unbiased for β.

Variances:

$$\hat{\beta}_1 - \beta = \frac{\sum_{i=1}^n X_i\varepsilon_i}{\sum_{i=1}^n X_i^2}.$$

$$\operatorname{Var}(\hat{\beta}_1) = \frac{\operatorname{Var}(\sum_{i=1}^n X_i\varepsilon_i)}{(\sum_{i=1}^n X_i^2)^2} = \frac{\sum_{i=1}^n X_i^2\sigma^2}{(\sum_{i=1}^n X_i^2)^2} = \frac{\sigma^2}{\sum_{i=1}^n X_i^2}.$$

$$\hat{\beta}_2 - \beta = \frac{\sum_{i=1}^n \varepsilon_i}{\sum_{i=1}^n X_i}.$$

$$\operatorname{Var}(\hat{\beta}_2) = \frac{\operatorname{Var}(\sum_{i=1}^n \varepsilon_i)}{(\sum_{i=1}^n X_i)^2} = \frac{n\sigma^2}{(\sum_{i=1}^n X_i)^2}.$$

Compare $\operatorname{Var}(\hat{\beta}_1)$ and $\operatorname{Var}(\hat{\beta}_2)$.

$$(\sum_{i=1}^n X_i)^2 \le n\sum_{i=1}^n X_i^2$$

(Cauchy--Schwarz inequality).

$$\Rightarrow \frac{1}{(\sum_{i=1}^n X_i)^2} \ge \frac{1}{n\sum_{i=1}^n X_i^2}.$$

$$\Rightarrow \operatorname{Var}(\hat{\beta}_2) = \frac{n\sigma^2}{(\sum_{i=1}^n X_i)^2} \ge \frac{n\sigma^2}{n\sum_{i=1}^n X_i^2} = \frac{\sigma^2}{\sum_{i=1}^n X_i^2} = \operatorname{Var}(\hat{\beta}_1).$$

Since the X_i are distinct and positive, the inequality is strict,

$$\Rightarrow \operatorname{Var}(\hat{\beta}_2) > \operatorname{Var}(\hat{\beta}_1).$$

Both estimators are unbiased and $\hat{\beta}_2$ has larger variance than $\hat{\beta}_1$.

Take $X_i = i^2, \qquad i = 1, 2, \dots, n.$

$$\sum_{i=1}^n X_i = \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$\sum_{i=1}^n X_i^2 = \sum_{i=1}^n (i^2)^2 = \sum_{i=1}^n i^4 = \frac{n(n+1)(2n+1)\big(3n^2+3n-1\big)}{30}.$$

$$\operatorname{Var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^n X_i^2} = \frac{\sigma^2}{\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}} = \frac{30\sigma^2}{n(n+1)(2n+1)(3n^2+3n-1)}.$$

$$\operatorname{Var}(\hat{\beta}_2) = \frac{n\sigma^2}{(\sum_{i=1}^n X_i)^2} = \frac{n\sigma^2}{\left(\frac{n(n+1)(2n+1)}{6}\right)^2} = \frac{36\sigma^2}{n(n+1)^2(2n+1)^2}.$$

Hence $\operatorname{Var}(\hat{\beta}_2) > \operatorname{Var}(\hat{\beta}_1)$ for all $n \ge 2$.

Thus, options C correct.

Q29 Text Solution:

Let $(X_1, \dots, X_k) \sim \text{Multinomial}(n; p_1, \dots, p_k), \quad p_i \ge 0, \ \sum_{i=1}^k p_i = 1.$

Then $X_1 + \cdots + X_k = n.$

Means:

$$E(X_i) = np_i, \quad i = 1, \dots, k.$$

Variances:

$$\operatorname{Var}(X_i) = np_i(1-p_i), \quad i = 1, \dots, k.$$

Covariances:

$$\operatorname{Cov}(X_i, X_j) = -np_ip_j, \quad i \ne j.$$

Variance--covariance matrix $\Sigma = \operatorname{Var}(X_1, \dots, X_k)$.

$$\Sigma_{ii} = np_i(1-p_i), \quad \Sigma_{ij} = -np_ip_j \ (i \ne j).$$

$$\Sigma = \begin{pmatrix} np_1(1-p_1) & -np_1p_2 & \cdots & -np_1p_k \\ -np_2p_1 & np_2(1-p_2) & \cdots & -np_2p_k \\ \vdots & \vdots & \ddots & \vdots \\ -np_kp_1 & -np_kp_2 & \cdots & np_k(1-p_k) \end{pmatrix}.$$

Compact matrix form:

$$p = (p_1, \dots, p_k)^\mathsf{T}, \quad \operatorname{Diag}(p) = \operatorname{diag}(p_1, \dots, p_k).$$

$$\Sigma = n\big(\operatorname{Diag}(p) - pp^\mathsf{T}\big).$$

Property 1: rows and columns sum to zero.

$$\Sigma\mathbf{1} = 0, \quad \mathbf{1} = (1, \dots, 1)^\mathsf{T}.$$

Hence the vector of ones is an eigenvector with eigenvalue 0.

Property 2: singularity and rank.

Since $\Sigma\mathbf{1} = 0$, Σ is singular.

$$\operatorname{rank}(\Sigma) = k - 1.$$

Property 3: signs of covariances.

$$\Sigma_{ii} = np_i(1-p_i) \ge 0, \quad \Sigma_{ij} = -np_ip_j \le 0 \ (i \ne j).$$

So variances are nonnegative and all off-diagonal covariances are nonpo

Property 4: correlation between X_i and X_j $(i \ne j)$.

$$\rho_{ij} = \frac{\operatorname{Cov}(X_i, X_j)}{\sqrt{\operatorname{Var}(X_i)\operatorname{Var}(X_j)}} = \frac{-np_ip_j}{\sqrt{np_i(1-p_i)}\sqrt{np_j(1-p_j)}}.$$

$$\rho_{ij} = -\sqrt{\frac{p_i p_j}{(1-p_i)(1-p_j)}}.$$

Property 5 (uniform case $p_i = 1/k$):

$$p_i = \frac{1}{k}\ \forall i \quad \Rightarrow \quad \Sigma = n\left(\frac{1}{k}I - \frac{1}{k^2}J\right),$$

where I is the $k \times k$ identity and J is the all-ones matrix.

Eigenvalues of $\Sigma:$ 0 (once), $\frac{n}{k}$ (multiplicity $k-1$).

So the largest eigenvalue is $\lambda_{\max} = \frac{n}{k}$.

We have : $k = 4,\ n = 50,\ p_1 = p_2 = p_3 = p_4 = \frac{1}{4}$.

$$E(X_i) = np_i = 50 \cdot \frac{1}{4} = 12.5, \quad i = 1,2,3,4.$$

$$\mathrm{Var}(X_i) = np_i(1-p_i) = 50 \cdot \frac{1}{4} \cdot \frac{3}{4} = 50 \cdot \frac{3}{16} = \frac{150}{16} = 9.375.$$

$$\mathrm{Cov}(X_i, X_j) = -np_ip_j = -50 \cdot \frac{1}{4} \cdot \frac{1}{4} = -50 \cdot \frac{1}{16} = -\frac{50}{16} = -3.125, \quad i \neq j.$$

Thus the covariance matrix Σ is

$$\Sigma = \begin{pmatrix} 9.375 & -3.125 & -3.125 & -3.125 \\ -3.125 & 9.375 & -3.125 & -3.125 \\ -3.125 & -3.125 & 9.375 & -3.125 \\ -3.125 & -3.125 & -3.125 & 9.375 \end{pmatrix}.$$

This matches the form $\Sigma = n\left(\frac{1}{4}I - \frac{1}{16}J\right) = 50\left(\frac{1}{4}I - \frac{1}{16}J\right)$.

Eigenvalues in the uniform case ($p_i = 1/4$) :

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = \lambda_4 = \frac{n}{k} = \frac{50}{4} = 12.5.$$

Hence $\mathrm{rank}(\Sigma) = 3, \quad \lambda_{\max} = 12.5$.

For $(X_1, \ldots, X_k) \sim \mathrm{Multinomial}(n; p_1, \ldots, p_k), \quad \Sigma = \mathrm{Var}(X) = n(\mathrm{Diag}(p) - pp^{\mathsf{T}})$.

$$\mathrm{tr}(\Sigma) = \sum_{i=1}^{k} \mathrm{Var}(X_i) = \sum_{i=1}^{k} np_i(1-p_i).$$

$$\mathrm{tr}(\Sigma) = n\sum_{i=1}^{k}(p_i - p_i^2) = n\left(1 - \sum_{i=1}^{k} p_i^2\right).$$

Uniform case: $p_1 = \cdots = p_k = \frac{1}{k}$.

$$\sum_{i=1}^{k} p_i^2 = k\left(\frac{1}{k}\right)^2 = \frac{1}{k}.$$

$$\therefore \quad \mathrm{tr}(\Sigma) = n\left(1 - \frac{1}{k}\right) = \frac{n(k-1)}{k}.$$

Given $k = 4,\ n = 50,\ p_i = \frac{1}{4}$.

$$\mathrm{tr}(\Sigma) = 50\left(1 - \frac{1}{4}\right) = 50 \cdot \frac{3}{4} = 37.5.$$

Since 0 is an eigen value of Σ.

eigenvalue of $\mathbf{2\Sigma^2 - \Sigma + I}$=2×0-0+1=1

Hence 1 is always an eigenvalue of $\mathbf{2\Sigma^2 - \Sigma + I}$.

Hence **C** is correct option.

Q30 Text Solution:

Solution:

X, Y, Z are i.i.d.Bernoulli(θ), $\quad 0 < \theta < 1$.

The joint pmf of X, Y, Z belongs to a one-parameter exponential family.

Hence $T = X + Y + Z$ is a natural sufficient statistic for θ.

(A) $(79e^X + 2024e^Y,\ 23\log(1+Z))$

Since $e^X, e^Y, \log(1+Z)$ are one--to--one functions of X, Y, Z,

the pair is a function of (X, Y, Z) and thus of $T = X + Y + Z$.

Therefore (A) is a sufficient statistic.

(B) e^{X+Y+Z}

The transformation $t \mapsto e^t$ is one--to--one,

so e^{X+Y+Z} is a minimal sufficient statistic (being a bijective function of

Hence (B) is true.

(C) $X + Y + Z$

The sum $T = X + Y + Z$ is the complete sufficient statistic in the Bernoulli family

(equivalently Binomial$(3, \theta)$ for T).

(Now,) $\quad 13(X + Y + Z)^2$

$T = X + Y + Z \sim \mathrm{Binomial}(3, \theta)$, which is a complete sufficient statistic for θ

.

The mapping $t \mapsto 13t^2$ is one--to--one on $\{0,1,2,3\}$,

hence $13(X+Y+Z)^2$ is a bijective function of T.

Therefore $13(X+Y+Z)^2$ is also complete and sufficient.

Thus (C) is true.

(D) $(X-2Y+Z)$

The distribution of $(X-2Y+Z)$ depends on θ, so it cannot be ancillary.

$$E(X)=E(Y)=E(Z)=\theta.$$

$$E(X-2Y+Z)=E(X)-2E(Y)+E(Z)=\theta-2\theta+\theta=0.$$

Although $E(X-2Y+Z)=0$, the full distribution depends on θ

since $P(X-2Y+Z=1)=P(X=1,Y=0,Z=0)=\theta(1-\theta)^2$,

and similarly for other values. Thus the law of $(X-2Y+Z)$

depends on θ.

Hence $(X-2Y+Z)$ is not ancillary.

Hence (D) is false.

Correct statements: (A), (B), (C).

Correct option (D).

Q31 Text Solution:

Let $\alpha \in \mathbb{R}$

Consider $x_n \to \alpha$

For option (a):

If $x_n \in \mathbb{Q}$

$\Rightarrow x_n = \frac{p_n}{q_n}$ if $\alpha \neq 0$

$\Rightarrow f(x_n) = \frac{\sin\frac{1}{q_n}}{\sin\frac{1}{p_n}} = \frac{p_n}{q_n}\frac{\frac{1}{p_n}}{\sin\frac{1}{p_n}}\frac{\sin\frac{1}{q_n}}{\frac{1}{q_n}}$

$= \frac{p_n}{q_n} \to \alpha$

Now if $x_n \in \mathbb{Q}^c$

$\Rightarrow f(x_n) = 1+\sin(x_n-1) \to 1+\sin(\alpha-1)$

Limit exist when $\alpha = 1+\sin(\alpha-1)$

$\Rightarrow \alpha-1 = \sin(\alpha-1)$

$x=\sin x \quad \text{iff } x=0$

$\Rightarrow \alpha-1=0$

$\Rightarrow \alpha=1$

$\Rightarrow S=\{1\}$.

Hence, option (A) and (B) are correct.

Q32 Text Solution:

Since 1 is the only eigen value of T and V is of dimension n.

So, characteristic polynomial is $(x-1)^n$.

By Cayley-Hamilton theorem

$(T-I)^n = 0 \Rightarrow (T-I)^{2n} = 0$

Hence, options (C) and (D) are correct.

Consider $n=3$ and T is an operator

such that $A = \begin{bmatrix} T \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$,

then $A-I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\left(A-I\right) \neq 0$, $\left(A-I\right)^2 \neq 0$.

Thus $\left(T-I\right) \neq 0$ and $\left(T-I\right)^2 \neq 0$.

Hence, options (A) and (B) are incorrect.

Q33 Text Solution:

$$\left[\begin{array}{cccccc|ccc} 2 & 0 & 3 & 2 & 0 & -2 & 5 & 5 & 0 \\ 0 & 1 & 0 & -1 & 3 & 4 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 4 & 4 & 1 & 3 & -2 \\ 1 & 1 & 1 & 0 & 1 & 1 & 4 & 3 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_4$

$$\sim \left[\begin{array}{cccccc|ccc} 1 & 1 & 1 & 0 & 1 & 1 & 4 & 3 & 1 \\ 0 & 1 & 0 & -1 & 3 & 4 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 4 & 4 & 1 & 3 & -2 \\ 2 & 0 & 3 & 2 & 0 & -2 & 5 & 5 & 0 \end{array}\right]$$

$R_4 \leftrightarrow R_4 - 2R_1$

$$\sim \left[\begin{array}{cccccc|ccc} 1 & 1 & 1 & 0 & 1 & 1 & 4 & 3 & 1 \\ 0 & 1 & 0 & -1 & 3 & 4 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 4 & 4 & 1 & 3 & -2 \\ 0 & -2 & 1 & 2 & -2 & -4 & -3 & -1 & -2 \end{array}\right]$$

$R_4 \leftrightarrow R_4 + 2R_2$

$$\left[\begin{array}{cccccc|ccc} 1 & 1 & 1 & 0 & 1 & 1 & 4 & 3 & 1 \\ 0 & 1 & 0 & -1 & 3 & 4 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 4 & 4 & 1 & 3 & -2 \\ 0 & 0 & 1 & 0 & 4 & 4 & -1 & 1 & -2 \end{array}\right]$$

$R_4 \leftrightarrow R_4 + 2R_1$

Thus we have

$Rank\,(M) = 3$

$\text{Rank}\,[M:b_1] = 4$

$\text{Rank}\,[M:b_2] = 4$

$\text{Rank}\,[M:\text{b}_1 - b_2] = 3$

So, system is constant for $b_1 - b_2$, but not for b_1 and b_2.

Q34 Text Solution:

Consider a trivariate discrete random variable $\mathbf{W} = (X, Y, Z)^t$ with pmf

$$f_{\mathrm{W}}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \left(\tfrac{1}{3}\right)^x \left(\tfrac{1}{6}\right)^y \left(\tfrac{1}{2}\right)^z \text{ with } x,\ y,$$

$z \in \{0,1\}$, $x + y + z = 1$

If we take random sample w_1 , w_2 , $\cdots$, w_{10} from $\mathbf{W}$. Define

$S_{10} = \frac{1}{10}$

$\sum_{i=1}^{10} (w_i - \overline{w})(w_i - \overline{w})^{\mathrm{T}}$

$P(X = 1) = P(X = 1\ ,\ Y = 0,\ Z = 0)$

$= \frac{1}{3}$

$P(X = 0) = 1 - \frac{1}{3} = \frac{2}{3}$

so, X~Ber $\left(\frac{1}{3}\right)$ then $1 - X$~Ber $\left(\frac{2}{3}\right)$

similarly ,

Y~Ber $\left(\frac{1}{6}\right)$

Z~Ber $\left(\frac{1}{2}\right)$

$\operatorname{Var}(Y^4) = \mathbb{E}(Y^8) - \left(\mathbb{E}(Y^4)^2\right)$

$\Rightarrow \frac{1}{6} - \left(\frac{1}{6}\right)^2 = \frac{5}{36}$

$\mathbb{E}\left(\begin{pmatrix} 100 \\ Z \end{pmatrix}\right) = \begin{pmatrix} 100 \\ 0 \end{pmatrix} \times \frac{1}{2} + \begin{pmatrix} 100 \\ 1 \end{pmatrix}$

$\times \frac{1}{2} = \frac{101}{2}$

$$S_{10} = \begin{bmatrix} S_X{}^2 & S_{XY} & S_{XZ} \\ S_{XY} & S_Y{}^2 & S_{YZ} \\ S_{XZ} & S_{YZ} & S_Z{}^2 \end{bmatrix}$$

$\operatorname{trace}(S_{10}) = S_X{}^2 + S_Y{}^2 + S_Z{}^2$

$\operatorname{trace}(\mathbb{E}(S_{10})) = \mathbb{E}(\operatorname{trace}(S_{10}))$

$\mathbb{E}\left(S_X{}^2 + S_Y{}^2 + S_Z{}^2\right)$

$= \frac{10}{9}\left\{\sigma_X{}^2 + \sigma_Y{}^2 + \sigma_Z{}^2\right\}$

$= \frac{10}{9}\left(1 - \left(\left(\frac{1}{3}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^2\right)\right) = \frac{110}{63}$

Hence correct option are (A) , (B) , (C).

Q35 Text Solution:

$$G_n = \left(\prod_{i=1}^n U_i\right)^{1/n} = \exp\left(\frac{1}{n}\sum_{i=1}^n \log U_i\right).$$

Let $V_i = \log U_i$. Then $G_n = \exp\left(\frac{1}{n}\sum_{i=1}^n V_i\right)$.

$$E[V_1] = \int_0^1 \log u\, du = [u \log u - u]_0^1 = -1.$$

$$\frac{1}{n}\sum_{i=1}^n V_i \xrightarrow{\text{a.s.}} E[V_1] = -1$$

(Strong Law of Large Numbers).

$$G_n = \exp\left(\frac{1}{n}\sum_{i=1}^n V_i\right) \xrightarrow{\text{a.s.}} e^{-1} = \frac{1}{e}.$$

Hence G_n converges almost surely (and thus in probability and in distri

Therefore G_n does not converge to 0 in any mode.

Thus statements A and B are false.

Since $G_n \xrightarrow{\text{a.s.}} \frac{1}{e}$, we have G_n

$\Rightarrow \frac{1}{e}$ (a degenerate X_2).

Hence statement C is true.

Moreover $0 < G_n \leq 1$ for all n, so for any $r > 0$, $|G_n - \frac{1}{e}|^r \leq M_r < \infty$.

By the Dominated Convergence Theorem, $E\left(|G_n - \frac{1}{e}|^r\right) \to 0$.

Hence G_n converges in r-th mean to X_2 for any $r > 0$.

Correct statements are C and D.

Q36 Text Solution:

(i) True. Since the sufficient statistic $|T|$ is a function of T, it follows that T is also a sufficient statistic.

(ii) True. Any function of a complete statistic is again a complete statist

(iii) True. If T^2 is sufficient, then T is sufficient (since T^2 is a function o

Since T is complete and sufficient, it follows that T is minimal sufficient

(iv) True. Consider $\psi(t) = t^3 - 2t^2 + 2t + 1$.

$$\psi'(t) = 3t^2 - 4t + 2 = 3\left(t - \frac{2}{3}\right)^2 + \frac{2}{9} > 0, \quad t \in \mathbb{R}.$$

Thus ψ is a one-to-one function, and hence preserves minimal sufficienc

Therefore $T^3 - 2T^2 + 2T + 1$ is a minimal sufficient statistic whenever T is.

Q37 Text Solution:

Solution:

Each X_i has density $f(x) = \dfrac{1}{\pi(1 + x^2)}$, $-\infty < x < \infty$, so $X_i \sim \text{Cauchy}(0, 1)$.

Result 1: If $X \sim \text{Cauchy}(0,1)$ **then** $aX \sim \text{Cauchy}(0, |a|)$.

Result 2: If $X \sim \text{Cauchy}(0, \gamma_1)$, $Y \sim \text{Cauchy}(0, \gamma_2)$ and independent, then $X + Y \sim \text{Cauchy}(0, \gamma_1 + \gamma_2)$.

(A) $2\sqrt{3}\,X_2 \sim \text{Cauchy}(0, 2\sqrt{3})$ by Result 1.

$$X_1 + 2\sqrt{3}X_2 \sim \text{Cauchy}(0, 1 + 2\sqrt{3}) \text{ by Result 2.}$$

$$\frac{X_1 + 2\sqrt{3}X_2}{\sqrt{3}} \sim \text{Cauchy}\left(0, \frac{1+2\sqrt{3}}{\sqrt{3}}\right) = \text{Cauchy}\left(0, 2 + \frac{1}{\sqrt{3}}\right).$$

Hence (A) is true.

Result 3: $\frac{1}{n}\sum_{i=1}^{n} X_i \sim \text{Cauchy}(0,1)$ for each n.

(B) $\frac{1}{n}\sum X_i$ always has the Cauchy(0, 1) distribution, hence does not converge in probability to 0. Therefore (B) is false.

Result 4: If a continuous distribution has unique median m, then the sample median of $\{X_1, \dots, X_{2n+1}\}$ converges in probability to m.

Since $F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan x$, $F(0) = \frac{1}{2}$, thus the median is 0.

(C) Hence median of $\{X_1, \dots, X_{2n+1}\}$ converges in probability to 0.

Thus (C) is true.

Result 5: For $X \sim \text{Cauchy}(0,1)$,

$$E(|X|^{\alpha}) = \frac{1}{\sin\left(\frac{\pi(\alpha+1)}{2}\right)}, \quad -1 < \alpha < 1.$$

(D) $E(|X_1|^{1/4}) = \frac{1}{\sin(5\pi/8)}$ by Result 5 with $\alpha = \frac{1}{4}$.

Hence (D) is true.

Therefore, (A), (C) and (D) are true and (B) is false.

Q38 Text Solution:

(a) One--dimensional marginals

$$P(Y_1 \le y) = P(\text{sign}(X_2)X_1 \le y) = \int_{-\infty}^{0} P(X_1 \ge y)\,\phi(u)\,du + \int_{0}^{\infty} P(X_1 \le y)\,\phi(u)\,du.$$

$$= \int_{-\infty}^{0} (1-\Phi(y))\phi(u)\,du + \int_{0}^{\infty} \Phi(y)\,\phi(u)\,du.$$

$$= (1-\Phi(y))\int_{-\infty}^{0} \phi(u)\,du \;+\; \Phi(y)\int_{0}^{\infty} \phi(u)\,du.$$

$$= (1-\Phi(y))\cdot\tfrac{1}{2} \;+\; \Phi(y)\cdot\tfrac{1}{2} = \Phi(y).$$

Thus $F_{Y_1}(y) = \Phi(y)$ i.e., $Y_1 \sim N(0,1)$.

Similarly $Y_2 \sim N(0,1)$ and $Y_3 \sim N(0,1)$.

$$F_{Y_1,Y_2}(y,z) = P(Y_1 \le y,\; Y_2 \le z) = P\big(\text{sign}(X_2)X_1 \le y,\; \text{sign}(X_3)X_2 \le z\big).$$

$$= P\big(\text{sign}(X_2)X_1 \le y,\; \text{sign}(X_3)X_2 \le z,\; X_3 \le 0\big) + P\big(\text{sign}(X_2)X_1 \le y,\; \text{sign}(X_3)X_2 \le z,\; X_3 > 0\big).$$

$$= P\big(\text{sign}(X_2)X_1 \le y,\; X_2 \ge -z,\; X_3 \le 0\big) + P\big(\text{sign}(X_2)X_1 \le y,\; X_2 \le z,\; X_3 > 0\big).$$

$$= \frac{1}{2}\,P\big(\text{sign}(X_2)X_1 \le y,\; X_2 \ge -z\big) + \frac{1}{2}\,P\big(\text{sign}(X_2)X_1 \le y,\; X_2 \le z\big),$$

(since $X_3 \perp (X_1, X_2)$).

Let $z \ge 0$.

$$P\big(\text{sign}(X_2)X_1 \le y,\; X_2 \ge -z\big) = P\big(X_1 \le y,\; X_2 \ge -z,\; X_2 \le 0\big) + P\big(-X_1 \le y,\; 0 \le X_2 < \infty\big).$$

$$= P(X_1 \le y)\,P(-z \le X_2 \le 0) + P(X_1 \ge -y)\,P(0 \le X_2 < \infty).$$

$$= \Phi(y)\big[\Phi(0) - \Phi(-z)\big] + \big[1 - \Phi(-y)\big]\big[1 - \Phi(0)\big].$$

$$= \Phi(y)\,\Phi(z).$$

Similarly $P\big(\text{sign}(X_2)X_1 \le y,\; X_2 \le z\big) = \Phi(y)\,\Phi(z)$.

If $z < 0$, one obtains the same final expression $\Phi(y)\Phi(z)$.

Combining both cases, we get $F_{Y_1,Y_2}(y,z) = \Phi(y)\Phi(z)$.

Hence $(Y_1, Y_2) \sim N_2\big((0,0), I_2\big)$.

(c) Joint distribution

The joint distribution of (Y_1, Y_2, Y_3) is not Gaussian.

$Y_1 > 0$ if $X_1 > 0,\ X_2 > 0$ or $X_1 < 0,\ X_2 < 0$.

$Y_1 < 0$ if $X_1 > 0,\ X_2 < 0$ or $X_1 < 0,\ X_2 > 0$.

$Y_2 > 0$ if $X_2 > 0,\ X_3 > 0$ or $X_2 < 0,\ X_3 < 0$.

$Y_2 < 0$ if $X_2 > 0,\ X_3 < 0$ or $X_2 < 0,\ X_3 > 0$.

$Y_3 > 0$ if $X_3 > 0,\ X_1 > 0$ or $X_3 < 0,\ X_1 < 0.$

$Y_3 < 0$ if $X_3 > 0,\ X_1 < 0$ or $X_3 < 0,\ X_1 > 0.$

Thus $P(Y_1 > 0,\ Y_2 > 0,\ Y_3 > 0) = P(X_1 > 0,\ X_2 > 0,\ X_3 > 0) + P(X_1 < 0,\ X_2 < 0,\ X_3 < 0) = \frac{1}{4}.$

$P(Y_i < 0,\ Y_j < 0,\ Y_k < 0) = 0$ for $i \neq j \neq k.$

Similarly, $P(Y_i > 0,\ Y_j < 0,\ Y_k < 0) = \frac{1}{4}$ for each $i \neq j \neq k.$

Hence the joint distribution of (Y_1, Y_2, Y_3) cannot be Gaussian.

Hence A, B, D are correct options.

Q39 Text Solution:

Solution.

$Z_i \sim N(4, 9), \qquad W_i \sim N(7, 16),$ independent for each i.

$$(A)\ \frac{1}{\sqrt{12}} \sum_{i=1}^{12} (Z_i - 4) \sim N(0,\ 9),$$

since it is a normalized sum of 12 i.i.d.$N(0, 9)$.

Hence statement (A) is true.

(B) Let $V_i = Z_i + W_i$. Then $E(V_i) = 11,$

$$\mathrm{Var}(V_i) = 9 + 16 + 2 \cdot \frac{1}{3} \cdot 3 \cdot 4 = 33.$$

$$\frac{V_i - 11}{\sqrt{33}} \sim N(0,1), \quad \sum_{i=1}^{12} (V_i - 11)^2 = 33\chi^2_{12},$$

$$\frac{1}{11} \sum_{i=1}^{12} (V_i - 11)^2 = 3\chi^2_{12},$$

which is a scaled chi-square, χ^2_{12}. Hence (B) is True.

(C) Put $U_i = \frac{Z_i - 4}{3} \sim N(0,1)$. Then $U_1 \perp (U_2, \ldots, U_{12})$.

$$\frac{\sqrt{11}\,(Z_1 - 4)}{\sqrt{\sum_{i=2}^{12} (Z_i - 4)^2}} = \frac{\sqrt{11} \cdot 3U_1}{\sqrt{\sum_{i=2}^{12} 9U_i^2}} = \sqrt{11}\ \frac{U_1}{\sqrt{\sum_{i=2}^{12} U_i^2}}.$$

$$\frac{U_1}{\sqrt{\frac{1}{11} \sum_{i=2}^{12} U_i^2}} \sim t_{11},$$

Hence (C) is true.

(D) Again put $V_i = \frac{W_i - 7}{4} \sim N(0,1)$.

$$\sum_{i=1}^{4} (W_i - 7)^2 = 16 \sum_{i=1}^{4} V_i^2 = 16\chi^2_4,$$

$$\sum_{i=5}^{12} (W_i - 7)^2 = 16\chi^2_8.$$

$$\frac{\sum_{i=1}^{4} (W_i - 7)^2/4}{\sum_{i=5}^{12} (W_i - 7)^2/8} = \frac{16\chi^2_4/4}{16\chi^2_8/8} = \frac{4\chi^2_4}{2\chi^2_8} = 2 \cdot \frac{\chi^2_4}{\chi^2_8} = 2F_{4,8},$$

so it is not exactly $F_{4,8}$. Hence (D) is false.

Therefore the correct statements are (A) , (B) and (C).

Q40 Text Solution:

Solution.

The likelihood function based on one observation $x > 1$ is $L(\theta) = 2\theta e^{-2\theta(x-1)}$.

The log--likelihood is $\ell(\theta) = \log 2 + \log \theta - 2\theta(x-1)$.

Differentiating, $\ell'(\theta) = \frac{1}{\theta} - 2(x-1)$.

Setting $\ell'(\theta) = 0$ gives $\hat{\theta} = \frac{1}{2(x-1)}$.

Thus statement (1) is true.

The maximum value of the likelihood is $L(\hat{\theta}) = 2\hat{\theta}\, e^{-2\hat{\theta}(x-1)} = \frac{1}{x-1} e^{-1}$.

$$\frac{1}{x-1} e^{-1} \neq \frac{2e^{-1}}{x-1},$$

so statement (2) is false.

The likelihood ratio statistic is $\Lambda(x) = \frac{L(2)}{L(\hat{\theta})} = \frac{2 \cdot 2\, e^{-4(x-1)}}{\frac{1}{x-1} e^{-1}} = 4(x-1)e^{-4x+5}$.

Since multiplication by a positive constant does not change the rejectio

$$\Lambda(x) \leq c \iff (x-1)e^{-4x} \leq k,$$

for some $k > 0$.

Hence a rejection rule of the form $2(x-1)e^{-2x} < k$ is consistent with the likelihood ratio test,

so statement (3) is true.

Consider $g(x) = (x-1)e^{-4x}$ for $x > 1$.

$g'(x) = e^{-4x}(1-(x-1)4) = e^{-4x}(5-4x)$

.

Thus g increases on $(1, \frac{5}{4})$ and decreases on $(\frac{5}{4}, \infty)$, so it is not monotone on $(1, \infty)$.

Therefore the rejection region cannot be of the form $x > c$ only.

Hence statement (4) is false.

$\boxed{\text{Correct statements are (1) and (3).}}$

Q41 Text Solution:

Given $a_n = \left(2^n + n2^n \sin^2 \frac{n}{2}\right)^{\frac{1}{2n - n\cos\frac{1}{n}}}$

$$\Rightarrow \log a_n = \frac{1}{2n - n\cos\frac{1}{n}} \log \left(2^n\left(1 + n\sin^2\frac{n}{2}\right)\right)$$

$$\Rightarrow \log a_n = \frac{\log 2^n}{2n - n\cos\frac{1}{n}} + \frac{\log\left(1 + n\sin^2\frac{n}{2}\right)}{2n - n\cos\frac{1}{n}}$$

Consider $b_n = \frac{n\log 2}{2n - n\cos\frac{1}{n}}$ and $c_n = \frac{\log\left(1+n\sin^2\frac{n}{2}\right)}{2n - n\cos\frac{1}{n}}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\log 2}{2 - \cos\frac{1}{n}} = \log 2$$

$$\& \quad |c_n| \le \left|\frac{\log(1+n)}{n}\right| \to 0$$

$\Rightarrow \lim_{n\to\infty} c_n = 0$ (By Squeeze theorem)

$$\Rightarrow \lim_{n\to\infty} \log a_n = \lim_{n\to\infty} b_n + \lim_{n\to\infty} c_n = \log 2 + 0$$

$$\Rightarrow \lim_{n\to\infty} a_n = 2$$

Q42 Text Solution:

$$a_n = \frac{n^5 \cdot n!}{5\cdot 6\cdot 7\cdot \ldots \cdot (5+n)} = \frac{1\cdot 2\cdot 3\cdot 4\cdot n^5 n!}{(5+n)!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot n^5 n!}{(n+5)(n+4)(n+3)(n+2)(n+1)n!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot n^5}{(n+5)(n+4)(n+3)(n+2)(n+1)}$$

$$\Rightarrow \lim_{n\to\infty} a_n = 4! = 24$$

Q43 Text Solution:

f is continuous at $x = a$ if and only if $\forall <a_n>, <b_n> \to a$

$$\Rightarrow \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(a)$$

Given $f(x) = \begin{cases} x^3 \sin\frac{1}{x} & x \in \mathbb{Q} \\ x^2 & x \in \mathbb{Q}^c \end{cases}$

If $<a_n> \to a; a_n \in \mathbb{Q}$
& $<b_n> \to a; b_n \in \mathbb{Q}^c$
$\lim_{n\to\infty} f(a_n) = a^3 \sin\frac{1}{a}$
and $\lim_{n\to\infty} f(b_n) = a^2$.

$$\Rightarrow a^3 \sin\frac{1}{a} = a^2$$

$$\Rightarrow a^2\left[a\sin\frac{1}{a} - 1\right] = 0$$

$$\Rightarrow a = 0$$

Hence, f is continuous at exactly one point.

Q44 Text Solution:

Solution.

$$U = X_1 + 2X_2, \qquad V = 3X_1 - X_2.$$

$$E[U] = \mu_1 + 2\mu_2, \qquad E[V] = 3\mu_1 - \mu_2.$$

$$\text{Var}(U) = \text{Var}(X_1) + 4\,\text{Var}(X_2) + 4\,\text{Cov}(X_1, X_2) = 1 + 4 + 4\rho = 5 + 4\rho.$$

$$\text{Var}(V) = 9\,\text{Var}(X_1) + \text{Var}(X_2) - 6\,\text{Cov}(X_1, X_2) = 9 + 1 - 6\rho = 10 - 6\rho.$$

$$\text{Cov}(U, V) = \text{Cov}(X_1 + 2X_2,\, 3X_1 - X_2)$$

$$= 3\,\text{Var}(X_1) - \text{Cov}(X_1, X_2) + 6\,\text{Cov}(X_2, X_1) - 2\,\text{Var}(X_2)$$

$$= 3 - \rho + 6\rho - 2 = 1 + 5\rho.$$

Hence the covariance matrix of (U, V) is

$$\begin{pmatrix} \text{Var}(U) & \text{Cov}(U,V) \\ \text{Cov}(U,V) & \text{Var}(V) \end{pmatrix} = \begin{pmatrix} 5+4\rho & 1+5\rho \\ 1+5\rho & 10-6\rho \end{pmatrix}.$$

For a bivariate normal (U, V),

$$\text{Var}(U \mid V) = \text{Var}(U) - \frac{\{\text{Cov}(U,V)\}^2}{\text{Var}(V)}.$$

$$\Rightarrow \text{Var}(U \mid V) = (5 + 4\rho) - \frac{(1+5\rho)^2}{10 - 6\rho}.$$

Now take

$$\rho = \frac{1}{\sqrt{5}}.$$

Then

$$5+4\rho = 5+\frac{4}{\sqrt{5}},$$

$$(1+5\rho)^2 = \left(1+\frac{5}{\sqrt{5}}\right)^2 = 6+\frac{10}{\sqrt{5}},$$

$$10-6\rho = 10-\frac{6}{\sqrt{5}}.$$

So

$$\mathrm{Var}(U \mid V) = \left(5+\frac{4}{\sqrt{5}}\right) - \frac{6+\frac{10}{\sqrt{5}}}{10-\frac{6}{\sqrt{5}}}.$$

This simplifies to

$$\mathrm{Var}(U \mid V) = \frac{196/5}{10-\frac{6}{\sqrt{5}}} = \frac{196}{5\left(10-\frac{6}{\sqrt{5}}\right)}.$$

Numerically, using $\rho = \frac{1}{\sqrt{5}} \approx 0.4472$,

$$5+4\rho \approx 6.7889, \quad 1+5\rho \approx 3.2361,$$

$$(1+5\rho)^2 \approx 10.4721, \quad 10-6\rho \approx 7.3167,$$

$$\frac{(1+5\rho)^2}{10-6\rho} \approx \frac{10.4721}{7.3167} \approx 1.43,$$

$$\mathrm{Var}(U \mid V) \approx 6.7889 - 1.43 \approx 5.36.$$

$$\boxed{\mathrm{Var}(X_1+2X_2 \mid 3X_1 - X_2) \approx 5.36.}$$

Q45 Text Solution:

Solution.

The cdf of X is

$$F(x) = \begin{cases} 0, & x < -1, \\ \frac{x+1}{4}, & -1 \le x < 0, \\ \frac{x+1}{3}, & 0 \le x < 1, \\ \frac{x+3}{6}, & 1 \le x < 2, \\ 1, & x \ge 2. \end{cases}$$

First we find the pdf and point masses from $F(x)$.

$$\text{For } -1 < x < 0: \quad f(x) = F'(x) = \frac{1}{4}.$$

$$\text{For } 0 < x < 1: \quad f(x) = F'(x) = \frac{1}{3}.$$

$$\text{For } 1 < x < 2: \quad f(x) = F'(x) = \frac{1}{6}.$$

Next we find jumps of F (x) (point masses).

$$P(X=0) = F(0) - \lim_{x\to 0^-} F(x) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

$$P(X=2) = F(2) - \lim_{x\to 2^-} F(x) = 1 - \frac{5}{6} = \frac{1}{6}.$$

There are no jumps at $x=-1$ or $x=1$, so no mass there.

Thus X has density $f(x) = \frac{1}{4}$ on $(-1,0)$, $\frac{1}{3}$ on $(0,1)$, $\frac{1}{6}$ on $(1,2)$,

and point masses $P(X=0) = \frac{1}{12}$, $P(X=2) = \frac{1}{6}$.

We now compute $E(e^{2X})$.

$$E(e^{2X}) = \int_{-1}^{0} e^{2x}\cdot\frac{1}{4}\,dx + \int_{0}^{1} e^{2x}\cdot\frac{1}{3}\,dx + \int_{1}^{2} e^{2x}\cdot\frac{1}{6}\,dx + \frac{1}{12}e^{2\cdot 0} + \frac{1}{6}e^{2\cdot 2}.$$

$$= \frac{1}{4}\int_{-1}^{0} e^{2x}\,dx + \frac{1}{3}\int_{0}^{1} e^{2x}\,dx + \frac{1}{6}\int_{1}^{2} e^{2x}\,dx + \frac{1}{12} + \frac{1}{6}e^4.$$

$$\int e^{2x}\,dx = \frac{1}{2}e^{2x}.$$

$$\frac{1}{4}\int_{-1}^{0} e^{2x}\,dx = \frac{1}{4}\cdot\frac{1}{2}(e^0 - e^{-2}) = \frac{1}{8}(1-e^{-2}).$$

$$\frac{1}{3}\int_{0}^{1} e^{2x}\,dx = \frac{1}{3}\cdot\frac{1}{2}(e^2-1) = \frac{1}{6}(e^2-1).$$

$$\frac{1}{6}\int_{1}^{2} e^{2x}\,dx = \frac{1}{6}\cdot\frac{1}{2}(e^4-e^2) = \frac{1}{12}(e^4-e^2).$$

So

$$E(e^{2X}) = \frac{1}{8}(1-e^{-2}) + \frac{1}{6}(e^2-1) + \frac{1}{12}(e^4-e^2) + \frac{1}{12} + \frac{1}{6}e^4.$$

We simplify by collecting like terms.

$$\text{Coefficient of } e^4: \quad \frac{1}{12}+\frac{1}{6} = \frac{1}{12}+\frac{2}{12} = \frac{1}{4}.$$

$$\text{Coefficient of } e^2: \quad \frac{1}{6}-\frac{1}{12} = \frac{2}{12}-\frac{1}{12} = \frac{1}{12}.$$

$$\text{Coefficient of } e^{-2}: \quad -\frac{1}{8}.$$

$$\text{Constant terms: } \frac{1}{8} - \frac{1}{6} + \frac{1}{12}.$$

$$\frac{1}{8} - \frac{1}{6} + \frac{1}{12} = \frac{3}{24} - \frac{4}{24} + \frac{2}{24} = \frac{1}{24}.$$

Hence

$$E(e^{2X}) = \frac{1}{4}e^4 + \frac{1}{12}e^2 - \frac{1}{8}e^{-2} + \frac{1}{24}.$$

Using $e^2 \approx 7.39,\ e^4 \approx 54.60,\ e^{-2} \approx 0.14,$

$$E(e^{2X}) \approx \frac{1}{4}\cdot 54.60 + \frac{1}{12}\cdot 7.39 - \frac{1}{8}\cdot 0.14 + \frac{1}{24}.$$

$$\approx 13.65 + 0.62 - 0.02 + 0.04 \approx 14.29.$$

$$\boxed{E(e^{2X}) \approx 14.29 \text{ (correct to two decimal places)}.}$$

Q46 Text Solution:

All X_i and Y_j are i.i.d. with density $f(x)$

$$= \frac{2}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right),\ x \geq 0.$$

Total sample size $n = 15 + 30 = 45$.

Let $z_1, \ldots,$ z_{45} denote all the X_i and Y_j together.

$$L(\sigma) = \prod_{k=1}^{45} \frac{2}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{z_k^2}{2\sigma^2}\right).$$

$$\ell(\sigma) = \log L(\sigma) = 45\log 2 - \frac{45}{2}\log(2\pi) - 45\log\sigma - \frac{1}{2\sigma^2}\sum_{k=1}^{45} z_k^2.$$

$$\frac{d\ell}{d\sigma} = -\frac{45}{\sigma} + \frac{1}{\sigma^3}\sum_{k=1}^{45} z_k^2.$$

$$\text{Set } \frac{d\ell}{d\sigma} = 0 \ \Rightarrow\ -\frac{45}{\sigma} + \frac{1}{\sigma^3}\sum_{k=1}^{45} z_k^2 = 0.$$

$$\Rightarrow \frac{1}{\sigma^3}\sum_{k=1}^{45} z_k^2 = \frac{45}{\sigma} \ \Rightarrow\ \sum_{k=1}^{45} z_k^2 = 45\sigma^2.$$

$$\Rightarrow \widehat{\sigma}^2 = \frac{1}{45}\sum_{k=1}^{45} z_k^2.$$

$$\sum_{k=1}^{45} z_k^2 = \sum_{i=1}^{15} X_i^2 + \sum_{j=1}^{30} Y_j^2 = 150 + 255 = 405.$$

$$\widehat{\sigma}^2 = \frac{405}{45} = 9 \ \Rightarrow\ \widehat{\sigma} = \sqrt{9} = 3.$$

Q47 Text Solution:

Solution:

The joint density of (X, Y) is

$$f(x,y) = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left[(x-\rho\cos\theta)^2 + (y-\rho\sin\theta)^2\right]\right\},$$
$$0 \leq \theta \leq 2\pi.$$

This is the density of a bivariate normal with independent components

$$X \sim N(\rho\cos\theta, 1), \qquad Y \sim N(\rho\sin\theta, 1), \qquad X \perp Y.$$

Hence the theoretical means are

$$E(X) = \rho\cos\theta, \qquad E(Y) = \rho\sin\theta.$$

Let the sample means be

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

By the method of moments we equate sample means to theoretical mea

$$\bar{X} = \rho\cos\theta, \qquad \bar{Y} = \rho\sin\theta.$$

$$\Rightarrow \frac{\bar{Y}}{\bar{X}} = \frac{\rho\sin\theta}{\rho\cos\theta} = \tan\theta.$$

Therefore the method of moments estimator of θ is

$$\hat{\theta}_{\text{MOM}} = \arctan\left(\frac{\bar{Y}}{\bar{X}}\right).$$

$$\bar{X} = \frac{100\sqrt{3}}{n}, \qquad \bar{Y} = \frac{300}{n}.$$

$$\frac{\bar{Y}}{\bar{X}} = \frac{300/n}{100\sqrt{3}/n} = \frac{300}{100\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

$$\hat{\theta}_{\text{MOM}} = \arctan(\sqrt{3}) = \frac{\pi}{3}.$$

$$\frac{27\alpha}{\pi}, \qquad \alpha = \frac{\pi}{3}$$

$$\frac{27\alpha}{\pi} = \frac{27\left(\frac{\pi}{3}\right)}{\pi} = \frac{27}{3}\cdot\frac{\pi}{\pi} = 9.$$

Q48 Text Solution:

$$\text{UMVUE of } P(X_1 < t) \text{ is } \hat{g}(\bar{X}) = \Phi\left(\frac{t-\bar{X}}{\sqrt{(n-1)/n}}\right).$$

Given $n = 5,\ \bar{X} = 1.08,\ t = 2,$ the UMVUE of $P(X_1 > 2)$ is

$$\widehat{P}(X_1 > 2) = \Phi\left(\frac{\bar{X}-2}{\sqrt{(n-1)/n}}\right).$$

$$= \Phi\left(\frac{1.08 - 2}{\sqrt{4/5}}\right).$$

$$= \Phi\left(\frac{-0.92}{2/\sqrt{5}}\right) = \Phi\left(\frac{-0.92}{0.8944}\right).$$

$$= \Phi(-1.0287) \approx 0.152.$$

Q49 Text Solution:

Let $X_i \sim N(\theta, 2),\ i = 1, 2.$

Define the standardized variable $Z_i = \dfrac{X_i - \theta}{\sqrt{2}} \sim N(0,1).$

We want: $P\left(X_{(1)} \le \theta \le X_{(2)}\right).$

This is the probability that at least one X_i is below θ and at least one is above:

$$P\left(X_{(1)} \le \theta \le X_{(2)}\right) = 1 - P(X_1 < \theta,\ X_2 < \theta) - P(X_1 > \theta,\ X_2 > \theta).$$

$$P(X_i < \theta) = P(Z_i < 0) = \frac{1}{2}.$$

$$P(\text{both} < \theta) = \left(\frac{1}{2}\right)^2 = \frac{1}{4},$$

$$P(\text{both} > \theta) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Hence the confidence coefficient is $1 - \dfrac{1}{4} - \dfrac{1}{4} = \dfrac{1}{2}.$

Q50 Text Solution:

We have the pdf

$$f_X(x) = \frac{\lfloor x \rfloor + 1}{6}, \quad 0 < x < 3.$$

Thus

$$f(x) = \frac{1}{6}\,(0 < x < 1),$$

$$f(x) = \frac{1}{3}\,(1 \le x < 2),$$

$$f(x) = \frac{1}{2}\,(2 \le x < 3).$$

We compute

$$p = F(2) = P(X \le 2).$$

On $(0,1)$ the probability is $\frac{1}{6}$, on $[1,2)$ it is $\frac{1}{3}$.

$$F(2) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}.$$

Hence $p = \frac{1}{2}$.

$$F_8(2) = \frac{1}{8}\sum_{i=1}^{8} I(X_i \le 2).$$

Each $I(X_i \le 2)$ is Bernoulli $\left(\frac{1}{2}\right)$, so

$$\sum_{i=1}^{8} I(X_i \le 2) \sim \text{Bin}\left(8, \frac{1}{2}\right).$$

Therefore

$$\text{Var}\left(\sum I(X_i \le 2)\right) = 8 \cdot \frac{1}{2} \cdot \frac{1}{2} = 2.$$

$$\text{Var}\left(F_8(2)\right) = \left(\frac{1}{8}\right)^2 \cdot 2 = \frac{1}{64} \cdot 2 = \frac{1}{32}.$$

$$\alpha = \frac{1}{32}.$$

$$32\alpha = 1.$$

Q51 Text Solution:

Given that, $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$\therefore A^2 = A \cdot A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$\therefore A^4 = A^2 \cdot A^2$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$\therefore A^8 = A^4 \cdot A^4$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$\therefore A^{16} = A^8 \cdot A^8$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 8 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^{32} = \begin{bmatrix} 1 & 0 & 0 \\ 16 & 1 & 0 \\ 16 & 0 & 1 \end{bmatrix}$$

$\therefore \quad A^{48} = A^{32} \cdot A^{16}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 24 & 1 & 0 \\ 24 & 0 & 1 \end{bmatrix}$$

$\therefore A^{50} = A^{48} \cdot A^2$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

Hence required trace is 3.

Q52 Text Solution:

Given $\lim_{x\to 0^+}\left(\sin kx+\cos kx+x\right)^{\frac{4}{x}}=e^{12}$

$\Rightarrow e^{\lim_{x\to 0^+}\left(\sin kx+\cos kx+x-1\right)\times\frac{4}{x}}=e^{12}$

$\Rightarrow e^{\lim_{x\to 0^+}4\left[\frac{\sin kx}{x}+\frac{\cos kx-1}{x}+1\right]}=e^{12}$

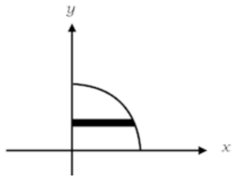
$\Rightarrow e^{\lim_{x\to 0^+}4\left[k\cos kx-k\sin kx+1\right]}=e^{12}$

$\Rightarrow e^{4\cdot(k+1)}=e^{12}$

$\Rightarrow k+1=3\Rightarrow k=2$

Q53 Text Solution:

To find $\iint_D \frac{xy}{\sqrt{1-y^2}}dxdy$; where D is the 1^{st} quadrant of the circle $x^2+y^2=1$.



In the first quadrant the limits are given by $0\le y\le 1,\ 0\le x\le\sqrt{1-y^2}$

$$\iint_D \frac{xy}{\sqrt{1-y^2}}dxdy$$
$$=\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}}\frac{xy}{\sqrt{1-y^2}}dxdy$$
$$=\int_{y=0}^{1}\frac{y}{\sqrt{1-y^2}}\left[\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}}dy$$
$$=\int_{y=0}^{1}\frac{y}{\sqrt{1-y^2}}\left[\frac{(1-y^2)}{2}\right]$$
$$=\frac{1}{2}\int_{y=0}^{1}y\sqrt{1-y^2}dy$$

Let $1-y^2=t$

$\Rightarrow\quad -2ydy=dt$

$\Rightarrow\quad \frac{1}{4}\int_0^1 t^{1/2}dt$

$\Rightarrow\quad \frac{1}{4}\frac{t^{3/2}}{3/2}\Big|_0^1$

$\Rightarrow\quad \frac{2\times 1}{4\times 3}=\frac{2}{12}=\frac{1}{6}$

Q54 Text Solution:

\textbf{Solution.}

We test $H_0: X\sim f_0$ against $H_A: X\sim f_1$.

The likelihood ratios are $\frac{f_1(1)}{f_0(1)}=0.5$, $\frac{f_1(2)}{f_0(2)}=0.6$, $\frac{f_1(3)}{f_0(3)}=0.8$, $\frac{f_1(4)}{f_0(4)}=0.9$, $\frac{f_1(5)}{f_0(5)}=\frac{0.72}{0.6}=1.2$.

Ordering from largest to smallest: $x=5,4,3,2,1$.

By the Neyman--Pearson lemma, the MP test rejects for large values of

Let the critical region be of the form ``reject H_0 when $X=5$'' with prol

The size condition under H_0 is $P_0(\text{reject})$
$=\gamma P_0(X=5)=\gamma\cdot 0.6=\alpha=0.2.$

$$\Rightarrow\qquad \gamma=\frac{0.2}{0.6}=\frac{1}{3}.$$

Thus the MP size 0.2 test is: reject H_0 if $X=5$ with probability $\frac{1}{3}$, otherwise do not reject.

The power under H_A is $P_1(\text{reject})=\gamma$
$P_1(X=5)=\frac{1}{3}\times 0.72=0.24.$

$\boxed{0.24}$

Q55 Text Solution:

We can use the Neyman–Pearson Lemma for specifying the rejection re

Let R represent the rejection region.

$$X\in R \text{ if } \frac{f(x|\theta_1)}{f(x|\theta_0)}>k$$

$$\frac{f(x|\theta_1)}{f(x|\theta=2)}=\frac{2^n\prod_{i=1}^{15}4x_i^3e^{-x_i/\theta_1}}{\theta_1^n\prod_{i=1}^{15}4x_i^3e^{-x_i/2}}$$
$$=\left(\frac{2}{\theta_1}\right)^n\exp\left\{\frac{\theta_1-2}{2\theta_1}\sum_{i=1}^{15}x_i^4\right\}$$

Since $\theta_1>2$, $\left(\frac{2}{\theta_1}\right)^n\exp$

$$\left\{\frac{\theta_1-2}{2\theta_1}\sum_{i=1}^{15}x_i^4\right\}>k\ \Leftrightarrow\ \sum_{i=1}^{15}x_i^4>c$$

$$R=\left\{X:\sum_{i=1}^{15}X_i^4>c\right\}$$

We should determine c in such a way that the size of the test equals

$$\alpha=0.05$$

$$P_{\theta_0}(X\in R)=\alpha\ \Rightarrow\ P_{\theta_0}\left(\sum_{i=1}^{15}X_i^4>c\right)=0.05$$

$$\frac{1}{\theta_0}\sum_{i=1}^{15}X_i^4\sim\Gamma(15,1)\ \text{ or }\ 2\frac{1}{\theta_0}\sum_{i=1}^{15}X_i^4\sim\chi^2_{(30)}$$

$$\text{Since } \theta_0=2,\quad \sum_{i=1}^{15}X_i^4\sim\chi^2_{(30)}$$

$$c=\chi^2_{30,\,0.05}=43.77$$

$$R = \left\{ X : \sum_{i=1}^{15} X_i^4 > 43.77 \right\}$$

Now the p-value is $\sup_{\theta \in \Theta_0} P_\theta$

$$\left(\sum_{i=1}^{15} X_i^4 > 46.98 \right) = P_{\theta=2}$$

$$\left(\sum_{i=1}^{15} X_i^4 > 46.98 \right) = 0.0249$$

Q56 Text Solution:

We have given X~ Dis Unif $(0, 1, 2, \ldots, 2026)$

U~U $[0, 1]$

Let $Y = X + U$

$P(Y = y) = P(X + U = y)$

$= P(X = [k], U = y - [k])$

Since X and U are independent .

$= P(X = [k]) P(U = y - [k])$

$\Rightarrow \frac{1}{2027} \times 1 = \frac{1}{2027}$, $0 \le y \le 2027$

So, Y~Unif $[0, 2027]$

then $Y - [Y]$~U $(0, 1)$

$Z = -\log(Y - [Y])$~exp (1)

Now $\mathbb{P}(Z > 2)$= 1- $\mathbb{P}(Z \le 2)$

$= e^{-2}$

$\therefore 512\, e^2 \mathbb{P}(Z > 2) = 512\, e^2 e^{-2} = 512$.

Q57 Text Solution:

Let a fair coin be tossed once. Then a fair die is rolled S times, where

$$S = \begin{cases} 120, & \text{if Head appears,} \\ 121, & \text{if Tail appears.} \end{cases}$$

Let X denote the total number of threes obtained. We want $P(\text{Head} \mid X = 30)$.

$$P(\text{Head} \mid X = 30) = \frac{P(X = 30 \mid \text{Head}) P(\text{Head})}{P(X = 30 \mid \text{Head}) P(\text{Head}) + P(X = 30 \mid \text{Tail}) P(\text{Tail})}.$$

$$P(\text{Head}) = P(\text{Tail}) = \frac{1}{2}.$$

$$P(\text{Head} \mid X = 30) = \frac{P(X = 30 \mid \text{Head})}{P(X = 30 \mid \text{Head}) + P(X = 30 \mid \text{Tail})}.$$

If Head occurs, $S = 120$ and X ~$\text{Bin}\left(120, \frac{1}{6}\right)$.

$$P(X = 30 \mid \text{Head}) = \binom{120}{30} \left(\frac{1}{6}\right)^{30} \left(\frac{5}{6}\right)^{90}.$$

If Tail occurs, $S = 121$ and X ~$\text{Bin}\left(121, \frac{1}{6}\right)$.

$$P(X = 30 \mid \text{Tail}) = \binom{121}{30} \left(\frac{1}{6}\right)^{30} \left(\frac{5}{6}\right)^{91}.$$

$$P(\text{Head} \mid X = 30) = \frac{\binom{120}{30}}{\binom{120}{30} + \binom{121}{30}\left(\frac{5}{6}\right)}.$$

$$\binom{121}{30} = \frac{121}{121 - 30} \binom{120}{30} = \frac{121}{91} \binom{120}{30}.$$

$$P(\text{Head} \mid X = 30) = \frac{1}{1 + \frac{121}{91} \cdot \frac{5}{6}} = \frac{546}{1151}.$$

$$\boxed{\frac{546}{1151}}$$

Q58 Text Solution:

We have given

$A = \left\{ (x, y) \in \mathbb{R}^2 : \ |x| - \frac{1}{3} < y < |x| + \frac{1}{3} \right\}$

$$f(x, y) = \begin{cases} \frac{1}{2} e^{-\frac{2}{3}|x-1|}, & (x, y) \in A, \\ 0, & \text{otherwise,} \end{cases}$$

Now ,

$$f(x, y) = \begin{cases} \underbrace{\frac{3}{2}}_{Y|X \sim U\left(|x| - \frac{1}{3};\, |x| + \frac{1}{3}\right)} \times \underbrace{\frac{1}{2} \times \frac{2}{3} e^{-\frac{2}{3}|x-1|}}_{X \sim L\left(1, \frac{3}{2}\right)}, & (x, y) \in A, \\ 0, & \text{otherwise,} \end{cases}$$

Therefore $E(Y|X) = |X|$

Now,

$\text{Cov}(X, Y) = \text{Cov}(X, E(Y|X))$

$= \text{Cov}(X, |X|)$

$\text{Cov}(X, |X|) = E(X|X|) - E(X)E(|X|)$

Let X~Laplace$\left(\mu = 1,\ b = \frac{3}{2}\right)$.

$f_X(x) = \frac{1}{2b} e^{-|x-\mu|/b} = \frac{1}{3} e^{-\frac{2}{3}|x-1|}, \quad x \in \mathbb{R}$.

$E(X|X|) = \int_{-\infty}^{\infty} x|x|\, f_X(x)\, dx.$

We split the integral at $x = 0$ and $x = 1$:

$E(X|X|) = \int_{-\infty}^{0} x|x| f(x)\, dx + \int_{0}^{1} x|x| f(x)\, dx + \int_{1}^{\infty} x|x| f(x)\, dx.$

For $x < 0$: $|x| = -x,\ |x - 1| = 1 - x,$

$\Rightarrow x|x| = -x^2.$

$I_1 = \int_{-\infty}^{0} x|x| f(x)\, dx = \int_{-\infty}^{0} \left(-x^2\right) \frac{1}{3} e^{-\frac{2}{3}(1-x)} dx = -\frac{1}{3} e^{-\frac{2}{3}} \int_{-\infty}^{0} x^2 e^{\frac{2}{3}x}\, dx.$

$\int_{-\infty}^{0} x^2 e^{\frac{2}{3}x}\, dx = \frac{27}{4}.$

$I_1 = -\frac{1}{3} e^{-\frac{2}{3}} \cdot \frac{27}{4} = -\frac{9}{4} e^{-\frac{2}{3}}.$

For $0 < x < 1$: $|x| = x,\ |x - 1| = 1 - x,$

$\Rightarrow x|x| = x^2.$

$I_2 = \int_0^1 x\left|x\right|f\left(x\right)\,dx =$

$\int_0^1 x^2 \frac{1}{3} e^{-\frac{2}{3}\left(1-x\right)}\,dx = \frac{1}{3}e^{-\frac{2}{3}} \int_0^1 x^2 e^{\frac{2}{3}x}\,dx.$

$\int_0^1 x^2 e^{\frac{2}{3}x}\,dx = \frac{15}{4}e^{\frac{2}{3}} - \frac{27}{4}.$

$I_2 = \frac{5}{4} - \frac{9}{4}e^{-\frac{2}{3}}.$

For $x > 1: \ |x| = x, \ |x-1| = x-1,$

$\Rightarrow x|x| = x^2.$

$I_3 = \int_1^\infty x\left|x\right|f\left(x\right)\,dx =$

$\int_1^\infty x^2 \frac{1}{3} e^{-\frac{2}{3}\left(x-1\right)}\,dx = \frac{1}{3}e^{\frac{2}{3}} \int_1^\infty x^2 e^{-\frac{2}{3}x}$ $dx.$

$\int_1^\infty x^2 e^{-\frac{2}{3}x}\,dx = \frac{51}{4}e^{-\frac{2}{3}}.$

$I_3 = \frac{1}{3}e^{\frac{2}{3}} \cdot \frac{51}{4}e^{-\frac{2}{3}} = \frac{17}{4}.$

$E\left(X\left|X\right|\right) = I_1 + I_2 + I_3$

$E\left(X\left|X\right|\right) = \frac{11}{2} - \frac{9}{2}e^{-\frac{2}{3}}.$

$\boxed{E\left(X\left|X\right|\right) = \frac{11}{2} - \frac{9}{2}e^{-2/3}}.$

Let X~Laplace$\left(\mu = 1, \ b = \frac{3}{2}\right).$

$f_X\left(x\right) = \frac{1}{2b}e^{-|x-\mu|/b} = \frac{1}{3}\,e^{-\frac{2}{3}|x-1|}, \quad x \in \mathbb{R}$

.

$E(|X|) = \int_{-\infty}^{\infty} |x|\, f_X(x)\,dx.$

Split the integral at $x = 0$ and $x = 1$.

$$E(|X|) = \int_{-\infty}^{0} |x| f_X(x)\,dx + \int_0^1 |x| f_X(x)\,dx + \int_1^\infty |x| f_X(x)\,dx.$$

For $x < 0: \ |x| = -x, \ |x-1| = 1-x.$

$$I_1 = \int_{-\infty}^{0} (-x)\frac{1}{3}\,e^{-\frac{2}{3}(1-x)}\,dx = -\frac{1}{3}e^{-\frac{2}{3}} \int_{-\infty}^{0} x\,e^{\frac{2}{3}x}\,dx.$$

$$I_1 = \frac{3}{2}e^{-\frac{2}{3}}.$$

For $0 < x < 1: \ |x| = x, \ |x-1| = 1-x.$

$$I_2 = \int_0^1 x\frac{1}{3}\,e^{-\frac{2}{3}(1-x)}\,dx = \frac{1}{3}e^{-\frac{2}{3}} \int_0^1 x e^{\frac{2}{3}x}\,dx.$$

$$I_2 = \frac{1}{2}\left(1 - e^{-\frac{2}{3}}\right).$$

For $x > 1: \ |x| = x, \ |x-1| = x-1.$

$$I_3 = \int_1^\infty x\frac{1}{3}\,e^{-\frac{2}{3}(x-1)}\,dx = \frac{1}{3}e^{\frac{2}{3}} \int_1^\infty x e^{-\frac{2}{3}x}\,dx.$$

$$I_3 = 1 + \frac{1}{2}e^{-\frac{2}{3}}.$$

$$E(|X|) = I_1 + I_2 + I_3 = \frac{3}{2}e^{-\frac{2}{3}} + \frac{1}{2}\left(1 - e^{-\frac{2}{3}}\right) + \left(1 + \frac{1}{2}e^{-\frac{2}{3}}\right).$$

$$E(|X|) = 1 + \frac{3}{2}e^{-\frac{2}{3}}.$$

$\boxed{E\left(|X|\right) = 1 + \frac{3}{2}\,e^{-2/3}}.$

$Cov\left(X, Y\right) = \frac{11}{2} - \frac{9}{2}e^{-\frac{2}{3}} - \left(1 + \frac{3}{2}e^{-\frac{2}{3}}\right) \times 1$

$= \frac{9}{2} - \frac{12}{2}e^{-\frac{2}{3}}$

$\alpha = \frac{9}{2}, \ \beta = -\frac{12}{2}, \ \gamma = -\frac{2}{3}$

$2\left(\alpha - \beta\right) - 3\gamma = 23$

Q59 Text Solution:

Given $X \sim U(0,2), \ Y \sim \text{Bernoulli}(\theta)$, independent, and $W = X + Y$.

Step 1. Distribution of W.

If $Y = 0$, then $W = X \in (0,2)$.

If $Y = 1$, then $W = X + 1 \in (1,3)$.

Because X is uniform on $(0,2)$, its density is $\frac{1}{2}$ on that interval, and zero outside.

Therefore

$$f_W(w) = \begin{cases} (1-\theta)\cdot\frac{1}{2}, & 0 < w < 2, \\ \theta\cdot\frac{1}{2}, & 1 < w < 3, \\ 0, & \text{otherwise.} \end{cases}$$

If $0 < w < 2$ then $Y = 0$ and hence $f_W(w) = (1-\theta)\cdot\frac{1}{2}.$

If $2 < w < 3$ then $Y = 1$ and hence $f_W(w) = \theta\cdot\frac{1}{2}.$

For the observations $0.3, \ 2.4, \ 0.8, \ 1.9, \ 2.1, \ 0.6$

the values in $(0,2)$ are $0.3, \ 0.8, \ 1.9, \ 0.6 \Rightarrow Y = 0$ four times.

The values in $(2,3)$ are $2.4, \ 2.1 \Rightarrow Y = 1$ two times.

Hence the likelihood is $L(\theta) \propto (1-\theta)^4\theta^2$.

$$\ell(\theta) = \ln L(\theta) = 4\ln(1-\theta) + 2\ln\theta.$$

$$\frac{d\ell}{d\theta} = \frac{2}{\theta} - \frac{4}{1-\theta} = 0.$$

$$2(1-\theta) = 4\theta.$$

$$2 = 6\theta.$$

$$\hat{\theta} = \frac{1}{3}.$$

We are asked for the MLE of θ^2.

$$\widehat{\theta^2} = (\hat{\theta})^2.$$

$$\widehat{\theta^2} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

$$\boxed{\widehat{\theta^2} = \frac{1}{9}}$$

$90\widehat{\theta^2} = 90 \times \frac{1}{9} \Rightarrow 10.$

Q60 Text Solution:

$$\text{Let } g(t) = \exp\left(\frac{3}{3+t}\right).$$

$$\text{Then } g(\overline{Y}) = \exp\left(\frac{3}{3+\overline{Y}}\right).$$

$$\text{We have } g'(t) = -\frac{3}{(3+t)^2}\exp\left(\frac{3}{3+t}\right).$$

$$\text{Hence } g'(\mu) = -\frac{3}{(3+\mu)^2}\exp\left(\frac{3}{3+\mu}\right).$$

$$\text{Assume } \sqrt{n}\,(\overline{Y}-\mu) \xrightarrow{D} N(0,\sigma^2).$$

$$\text{By the delta method, } \sqrt{n}\left(g(\overline{Y})-g(\mu)\right) \xrightarrow{D} N\left(0,\ [g'(\mu)]^2\sigma^2\right).$$

$$[g'(\mu)]^2 = \frac{9}{(3+\mu)^4}\exp\left(\frac{6}{3+\mu}\right).$$

$$\boxed{\sqrt{n}\left(g(\overline{Y})-g(\mu)\right) \xrightarrow{D} N\left(0,\ \frac{9\sigma^2}{(3+\mu)^4}\exp\left(\frac{6}{3+\mu}\right)\right)}$$

Now ,

$$\text{Let } Y \sim \text{NB}(r=3, p=\tfrac{1}{3}),\ \text{i.e. } P(Y=y) = \binom{y+2}{2}\left(\tfrac{1}{3}\right)^3\left(\tfrac{2}{3}\right)^y,\ y=0,1,2,\ldots$$

For this parametrization (number of failures until r successes),

$$\mu = E(Y) = \frac{r(1-p)}{p} = 3\cdot\frac{2/3}{1/3} = 6,$$

$$\sigma^2 = \text{Var}(Y) = \frac{r(1-p)}{p^2} = 3 \cdot \frac{2/3}{(1/3)^2} = 18.$$

Let $\overline{Y}$ be the sample mean of an i.i.d. sample from this NB distribution.

$$\text{By the CLT, } \sqrt{n}\,(\overline{Y}-6) \xrightarrow{D} N(0,18).$$

$$\text{Now take } g(t) = \exp\left(\frac{3}{3+t}\right),$$

$$g(\overline{Y}) = \exp\left(\frac{3}{3+\overline{Y}}\right).$$

$$g'(t) = -\frac{3}{(3+t)^2}\exp\left(\frac{3}{3+t}\right),$$

$$g'(6) = -\frac{3}{9^2}\exp\left(\frac{3}{9}\right) = -\frac{3}{81}e^{1/3}.$$

$$[g'(6)]^2 = \frac{9}{9^4}e^{2/3} = \frac{9}{6561}e^{2/3} = \frac{1}{729}e^{2/3}.$$

$$\text{By the delta method, } \sqrt{n}\left(g(\overline{Y})-g(6)\right) \xrightarrow{D} N\left(0,\ [g'(6)]^2\sigma^2\right).$$

$$[g'(6)]^2\sigma^2 = \frac{1}{729}e^{2/3}\cdot 18 = \frac{18}{729}e^{2/3} = \frac{2}{81}e^{2/3}.$$

$$\boxed{\sqrt{n}\,\left(g(\overline{Y})-g(6)\right) \xrightarrow{D} N\left(0,\ \frac{2}{81}e^{2/3}\right)}$$